Tutorial 1: Groups and symmetry

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1 Groups

Question: What is a group?

A group is a set of objects (*elements* or *members*), which can be combined by some operation (addition, multiplication, etc.). For a description of the Standard Model (SM), it is only necessary to consider groups under multiplication, and therefore the notation of multiplication will be used for simplicity from the start. When applying this operation, four conditions must be satisfied:

1. For all elements a, b in the group, the combination ab is also a member of the group.
2. The operation must be associative, i.e. $(ab)c = a(bc)$.
3. There is an identity element e, such that $ae = ea = a$ for all elements.
4. Every element a has an inverse a^{-1}, such that $aa^{-1} = a^{-1}a = e$.

The identity element is commonly written e for generality. For multiplicative groups, e is just the number 1, or an appropriate identity matrix. Under addition, for example, the identity element is 0. From now on, "1" will replace e as labeling the identity element.

1.1 Examples

The simplest group is the trivial group:

$$\{1\}. \tag{1}$$

This has only one member, and yet satisfies all of the properties required of a group (under multiplication). We can add one member to this to construct a simple non-trivial group:

$$\{1, -1\}. \tag{2}$$

Here, each element is its own inverse.

We can further extend this, to construct a four-element complex group:

$$\{1, -1, i, -i\}. \tag{3}$$

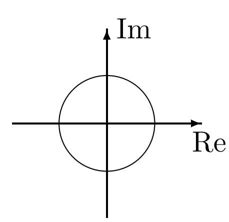
The new elements, i and $-i$, are inverses of each other. Although still a very simple group to analyse, it is instructive to write out the multiplication table for this group.

	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

The top-left section of this table involves only the elements 1 and -1. In fact, the group of Equation (2) is a *subgroup* of Equation (3). The bottom-right corner is not equivalent - the i and $-i$ elements do *not* form a subgroup. However, if they did, then this group would factorise into a *product* of groups. In the Standard Model, this is exactly what happens; the final symmetry group of the SM is a product of three simpler groups, $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$. These component subgroups are, however, more complex than the simple discrete groups considered so far. In fact, they all belong to a category of continuous groups, known as *Lie groups*.

1.2 Lie groups

The defining property of a Lie group is that all elements can be reached by successive infinitesimal steps, usually starting from the identity element. Here we introduce the group U(1) to serve as an example. The name U(1) derives from the fact that this is a *unitary* group with one dimension. All numbers on the complex plane with modulus 1 are members of this group.



Consider a member $(1 + i\epsilon)$ of this group a small distance ϵ from the identity. When thought of in terms of transformations, this corresponds to a small (eventually, infinitesimal) rotation of the complex plane. Rewriting ϵ as α/N, where N is a large integer, we can imagine applying this small rotation N times. Mathematically, we achieve this by multiplying $(1+i\alpha/N)$

by itself N times, i.e. by computing $(1 + i\alpha/N)^N$. Taking the limit as $N \to \infty$, we obtain a generic member of the group:

$$\lim_{N\to\infty}\left(1 + i\frac{\alpha}{N}\right)^N = e^{i\alpha}. \tag{4}$$

This is therefore a Lie group, as only an infinitesimally small region around the identity element needs to be known in order to characterise the entire group.

Due to this property, Lie groups are often characterised in terms of their *generators*, which in our case can be thought of as unit vectors describing possible directions in which transformations can be made. The number of these directions is called the *dimension* n of the group. With a collection of generators $\boldsymbol{T}$, and associated parameters $\boldsymbol{\alpha}$ (again, one for each dimension), a generic Lie group member is written in exponential notation like Equation (4):

$$\lim_{N\to\infty}\left(1 + i\frac{\boldsymbol{\alpha}\cdot\boldsymbol{T}}{N}\right)^N = e^{i\boldsymbol{\alpha}\cdot\boldsymbol{T}}. \tag{5}$$

U(1) has only one dimension, and also only one generator. Comparing Equations (4) and (5), it is clear that the generator for U(1) is simply the number 1. When there is more than one generator, it should be remembered that the properties of groups say nothing about whether or not different members of the group will *commute*. The same observation applies to the generators of a Lie group. The commutators of the group generators define the *algebra* of the group:

$$[T_a, T_b] = T_aT_b - T_bT_a = if_{abc}T_c. \tag{6}$$

Note that the right-hand side is linear in the group generators; this is a consequence of all products of group members also being in the group.

The numbers f_{abc} are called the group's *structure constants*. If all are zero, then the group's generators (and elements) all commute; the group is *Abelian*. U(1) is an Abelian group, as all complex numbers with modulus one commute with each other. Multi-dimensional groups may be *non-Abelian*. When applied to field theories, these groups lead to self-interacting gauge fields.

2 Gauge transformations and symmetry groups

2.1 Preamble: Classical electromagnetism

The classical Lagrangian for an otherwise free particle of charge Q in an electromagnetic field is

$$L = \frac{1}{2}m\dot{\boldsymbol{x}}^2 + Q\dot{\boldsymbol{x}}\cdot\boldsymbol{A} - Q\Phi. \tag{7}$$

Using this, one finds that the canonical momentum $\boldsymbol{p} = \partial L/\partial \dot{\boldsymbol{x}}$ is no longer the physical particle's momentum $m\dot{\boldsymbol{x}}$, but has an additional contribution from the electromagnetic field:

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{x}}} = m\dot{\boldsymbol{x}} + Q\boldsymbol{A} \tag{8}$$

Similarly, the classical Hamiltonian has a contribution from the electric potential $Q\Phi$

$$\begin{aligned} H &= \boldsymbol{p}\cdot\dot{\boldsymbol{x}} - L \\ &= (m\dot{\boldsymbol{x}} + Q\boldsymbol{A})\cdot\dot{\boldsymbol{x}} - \frac{1}{2}m\dot{\boldsymbol{x}}^2 - Q\dot{\boldsymbol{x}}\cdot\boldsymbol{A} + Q\Phi \\ &= \frac{1}{2}m\dot{\boldsymbol{x}}^2 + Q\Phi. \end{aligned} \tag{9}$$

Formally, we can then recover the particle's physical energy and momentum from the canonical variables by subtracting these extra contributions:

$$\begin{aligned} \boldsymbol{p}_{\text{phys.}} &= m\dot{\boldsymbol{x}} = \boldsymbol{p} - Q\boldsymbol{A} \\ E_{\text{phys.}} &= \frac{1}{2}m\dot{\boldsymbol{x}}^2 = H - Q\Phi. \end{aligned} \tag{10}$$

These substitutions have exact analogues in all forms of quantum mechanics, including field theory, when one wishes to incorporate electromagnetic effects on a particle's motion. Naturally, classical variables must be replaced by appropriate quantum mechanical operators, but the form of the substitutions in Equation (10) is identical. As we will later see, the other forces in the Standard Model can be introduced using similar modifications of canonical variables.

Before moving on to considering quantum mechanical operators, one should remember that the EM potentials Φ and $\boldsymbol{A}$ can be freely modified by a *gauge transformation*, with ultimately no physical effects:

$$\Phi \to \Phi' = \Phi - \frac{\partial \chi}{\partial t}; \qquad \boldsymbol{A} \to \boldsymbol{A'} = \boldsymbol{A} + \nabla\chi. \tag{11}$$

Here, χ is understood to be an arbitrary function of space and time. Note that this will change the canonical variables defined in Equations (7), (8) and (9), but *not* the physical momenta calculated in Equation (10).

2.2 Electromagnetism in quantum mechanics

In the quantum mechanical context, canonical energy and momentum variables are replaced by differential operators in the usual way:

$$\hat{E} = i\frac{\partial}{\partial t} \text{ and } \hat{\boldsymbol{p}} = -i\nabla; \text{ or, in covariant notation } \hat{p}_\mu = i\partial_\mu. \tag{12}$$

Equations of motion inevitably involve these operators acting on wavefunctions or fields, here referred to generically with the symbol ψ. In addition, covariant notation will be used for simplicity, although the separation into energy and momentum components (e.g. for application in the Schrödinger equation) is straightforward.

When electromagnetic interactions are included, physical four-momentum operators can be extracted by modifying the canonical operators in a way analagous to Equation (10):

$$\begin{aligned} \hat{p}_\mu &\rightarrow \hat{p}_\mu - QA_\mu \\ &= i(\partial_\mu + iQA_\mu) \\ &= i\mathcal{D}_\mu \end{aligned} \tag{13}$$

In the last line, the *covariant derivative* $\mathcal{D}_\mu = \partial_\mu + iQA_\mu$ is introduced, which replaces ∂_μ when calculating physical measurables for interacting particles or fields.

One immediate issue is that the covariant derivative varies under a gauge transformation (11) to

$$\mathcal{D}_\mu \rightarrow \mathcal{D}'_\mu = \partial_\mu + iQA_\mu - iQ\partial_\mu\chi \tag{14}$$

Thus, it appears that the physical measurable expectation values $\langle\psi|\hat{E}|\psi\rangle$ and $\langle\psi|\hat{\boldsymbol{p}}|\psi\rangle$ change, violating gauge invariance. This apparent problem can be elegantly solved by additionally transforming the wavefunction ψ, which has no classical analogue, according to

$$\psi \rightarrow \psi' = e^{iQ\chi}\psi. \tag{15}$$

Then, the combination $\mathcal{D}_\mu\psi$ transforms as follows:

$$\begin{aligned} \mathcal{D}_\mu\psi \rightarrow \mathcal{D}'_\mu\psi' &= \partial_\mu(e^{iQ\chi}\psi) + ie^{iQ\chi}QA_\mu\psi - iQ(\partial^\mu\chi)e^{iQ\chi}\psi \\ &= e^{iQ\chi}\partial_\mu\psi + iQ(\partial_\mu\chi)e^{iQ\chi}\psi + ie^{iQ\chi}QA_\mu\psi - iQ(\partial^\mu\chi)e^{iQ\chi}\psi \\ &= e^{iQ\chi}(\partial_\mu + iQA_\mu)\psi \\ &= e^{iQ\chi}\mathcal{D}_\mu\psi, \end{aligned} \tag{16}$$

modified only by an overall (space-time-dependent) phase. Any physical expectation value will be pre-multiplied by the conjugate of a wave function which transforms according to $\psi^* \rightarrow \psi^{*\prime} = e^{-iQ\chi}\psi^*$, and will therefore be left unchanged by the *combined* transformation described by Equations (14) and (15).

Exercise: Prove that this also works for the second derivative $\mathcal{D}_\mu\mathcal{D}^\mu\psi$, as in the Klein-Gordan equation, for example.

Exercise: Beginning with a free particle ($A^\mu = 0$), consider the gauge transformation given by $\chi = -\arg(\psi)/Q$. What are the resulting potentials and new wavefunction? Interpret the result in terms of physical quantities.

Exercise: (more tricky) Repeat the previous exercise for $\chi = -\frac{\arg(\psi)}{Q(1+e^{-t})}$.

2.3 Relation to the U(1) symmetry group

It is well known that all observable quantites are invariant under a global phase transformation of a wave function or field

$$\psi \to \psi' = e^{i\phi}\psi, \tag{17}$$

where the phase ange ϕ does not depend on space-time coordinates. The prefactor $e^{i\phi}$ can be recognised as a member of the U(1) group considered in Section 1.2. The transformation (17) is referred to as a *global* U(1) *transformation*. Note that it alters an *internal* space of the wavefunction, distinct from external space-time.

When electromagnetic interactions are introduced, we find that now a much more stringent symmetry exists, that of a *local* U(1) transformation, of Equation (15) (recall that χ is an *arbitrary* function of space-time). Thus, it appears that internal symmetries of a wavefunction or field are deeply related to their gauge interactions. If we began with a non-interacting (free) particle, we could "derive" the electromagnetic interaction by requiring that the transformation in Equation (15) has no effect on measurable quantities. We would then be forced to introduce a new field A^μ that simultaneously transforms as Equation (11), and to form a covariant derivative like (13) to represent the particle's physical four-momentum.

In the Standard Model, all gauge interactions are derived in this way, starting from Lie group operators applied to an internal space (called a Hilbert space), and demanding symmetry in the physical equations upon arbitrary local rotations within this space. Unlike electromagnetism, the other gauge groups of the SM are non-Abelian, a feature we will begin to explore in the next section.

2.4 A first look at SU(2)

The name SU(2) refers to the group of *special* unitary matrices with two dimensions. The term "special" means that these matrices have determinant 1, thus preserving the normalisation of state vectors upon which they act. As with any Lie group, any member of SU(2), G, can be written in terms of the group's generators $\boldsymbol{T}$

$$G = e^{i\boldsymbol{\alpha}\cdot\boldsymbol{T}}. \tag{18}$$

The generators of SU(2) are familiar, as they are proportional to the Pauli spin matrices:

$$T_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad T_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad T_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{19}$$

These are Hermitian matrices with zero trace, which ensure that the group elements in Equation (18) are unitary with unit determinant. Correspondingly, $\boldsymbol{\alpha}$ is a three-component vector in the associated Hilbert space. In addition, the generator matrices have the following important properties:

$$[T_a, T_b] = i\epsilon_{abc}T_c, \text{ and } T_a^2 = \frac{1}{2}\boldsymbol{I}. \tag{20}$$

Exercise: Express $\boldsymbol{\alpha} \cdot \boldsymbol{T}$ as a 2×2 matrix. Use the Taylor series expansion of (18) to find G. Does it have the expected properties? What value of $|\boldsymbol{\alpha}|$ corresponds to a full rotation?

Assuming that these matrices can operate on a wavefunction ψ, we can begin to deduce the form of the associated gauge fields by insisting on a *local* symmetry based on this group. That is, we are allowed to vary $\boldsymbol{\alpha}$ arbitrarily in space and time. The state ψ transforms as follows:

$$\psi \to \psi' = G\psi, \tag{21}$$

and we introduce a covariant derivative to account for the interactions with this field:

$$\mathcal{D}_\mu = \partial_\mu + igB_\mu = \partial_\mu + ig\boldsymbol{T} \cdot \boldsymbol{b}_\mu. \tag{22}$$

$\boldsymbol{b}_\mu$ is a vector of three new gauge fields, corresponding to the number of group generators. g is an associated coupling strength, analagous to the electric charge. Under the gauge transformation G, the field B_μ transforms as follows:

$$B_\mu \to B'_\mu = GB_\mu G^{-1} + \frac{i}{g}(\partial_\mu G)G^{-1}. \tag{23}$$

Exercise: Show that under this transformation $\mathcal{D}'_\mu\psi' = G\mathcal{D}_\mu\psi$.

Equation (23) is the general form for the transformation of a gauge field derived through symmetry principles. For Abelian groups, Equation (23) reduces to the same form as given in Equation (11) for the U(1) transformation. As SU(2) is a non-Abelian group, G and B_μ do not commute, which complicates further analysis but also yields a rich structure for the gauge fields. This will be examined further in the next tutorial.