Tutorial 2: Non-Abelian groups

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1 Introduction: from tutorial 1

In the first tutorial, we saw that a generic SU(2) transformation may be written as

$$G = e^{i\boldsymbol{\alpha}\cdot\boldsymbol{T}}, \tag{1}$$

where $\boldsymbol{T}$ is a vector of group generators (three 2×2 matrices in the case of SU(2)), and $\boldsymbol{\alpha}$ is a vector of parameters describing the transformation. We then allow a wave function (or field) ψ to undergo a *local* transformation $\psi \to \psi' = G\psi$, meaning that $\boldsymbol{\alpha}$ varies arbitrarily in space and time. We can ensure that our equations[1] are invariant under this arbitrary transformation by introducing a field $B_\mu = \boldsymbol{T} \cdot \boldsymbol{b}_\mu$ into a covariant derivative:

$$\mathcal{D}_\mu = \partial_\mu + igB_\mu = \partial_\mu + ig\boldsymbol{T} \cdot \boldsymbol{b}_\mu. \tag{2}$$

Thus, B_μ contains a number of component fields $\boldsymbol{b}_\mu$ equal to the number of generators, *i.e.* the dimension of the group. We then require that B_μ undergoes a simultaneous gauge transformation according to

$$B_\mu \to B'_\mu = GB_\mu G^{-1} + \frac{i}{g}(\partial_\mu G)G^{-1}. \tag{3}$$

This formalism can describe all the gauge groups of the Standard Model: U(1) electromagnetism (one generator, one gauge field), SU(2) for the weak nuclear force (three gauge fields) and SU(3) for the strong nuclear force (eight gauge fields).

2 Non-Abelian groups in gauge theories

When we considered U(1), the analysis of the gauge transformation was considerably simplified by the fact that the group was Abelian. In fact, there is only one U(1) generator, which trivially commutes with itself. The generators of the other symmetries involved in Standard Model forces, do

[1] This really means all physical observables.

not commute. In other words, not all structure constants f_{abc} are zero, where f_{abc} is defined by

$$[T_a, T_b] = T_a T_b - T_b T_a = i f_{abc} T_c. \tag{4}$$

For SU(2), the structure constants are equal to the completely antisymmetric Levi-Civita symbol ϵ_{abc}. In the case of SU(3), the constants are more complex, and given in the lecture notes.

To examine the immediate consequences of non-commutation of the generators, consider Equation (3) written in terms of the component fields

$$\begin{aligned} \boldsymbol{T} \cdot \boldsymbol{b}'_\mu &= e^{i\boldsymbol{\alpha}\cdot\boldsymbol{T}} \boldsymbol{T} \cdot \boldsymbol{b}_\mu e^{-i\boldsymbol{\alpha}\cdot\boldsymbol{T}} + \frac{i}{g}(\partial_\mu e^{i\boldsymbol{\alpha}\cdot\boldsymbol{T}}) e^{-i\boldsymbol{\alpha}\cdot\boldsymbol{T}} \\ \text{or } T_a b'_{\mu,a} &= e^{i\alpha_b T_b} b_{\mu,a} T_a e^{-i\alpha_c T_c} + \frac{i}{g}(\partial_\mu e^{i\alpha_a T_a}) e^{-i\alpha_b T_b} \end{aligned} \tag{5}$$

In the second version, the Einstein summation convention for indices has been assumed. This describes a general transformation, but often we will be interested in the perturbative regime, when the transformation G is close to the identity. In this case, $|\boldsymbol{\alpha}|$ is small, and $e^{i\boldsymbol{\alpha}\cdot\boldsymbol{T}} \approx 1 + i\boldsymbol{\alpha}\cdot\boldsymbol{T}$. Under this infinitesimal transformation, Equation (5) becomes the following (dropping any second-order terms in $\boldsymbol{\alpha}$ and $\partial_\mu \boldsymbol{\alpha}$):

$$\begin{aligned} T_a b'_{\mu,a} &\simeq (1 + i\alpha_b T_b) b_{\mu,a} T_a (1 - i\alpha_c T_c) - \frac{1}{g}(\partial_\mu \alpha_a) T_a (1 - i\alpha_b T_b) \\ &= b_{\mu,a} T_a - i b_{\mu,a}(\alpha_c T_a T_c - \alpha_b T_b T_a) - \frac{1}{g}(\partial_\mu \alpha_a) T_a + \mathcal{O}(\boldsymbol{\alpha}^2) \end{aligned} \tag{6}$$

Here, we note that the indices b and c on the right hand side are arbitrary, so we can rewrite $\alpha_c T_a T_c$ as $\alpha_b T_a T_b$ to obtain

$$\begin{aligned} T_a b'_{\mu,a} &= b_{\mu,a} T_a - i\alpha_b b_{\mu,a} [T_a, T_b] - \frac{1}{g}(\partial_\mu \alpha_a) T_a \\ &= b_{\mu,a} T_a + f_{abc} \alpha_b b_{\mu,a} T_c - \frac{1}{g}(\partial_\mu \alpha_a) T_a \end{aligned} \tag{7}$$

Exercise: Find an expression for $b'_{\mu,a}$ that will always satisfy Equation (7). Make sure it reduces to the electromagnetic gauge transformation in the case of a U(1) symmetry.

Equation (7) shows how B_μ transforms under an infinitesimal gauge transformation, but we have not yet shown that physical observables are unchanged. By analogy with the U(1) case, it is enough to show that $\mathcal{D}'_\mu \psi' = G \mathcal{D}_\mu \psi$, as the conjugate part of any expectation value will always cancel the G on the right hand side. This condition can be easily verified, again to

$\mathcal{O}(\boldsymbol{\alpha})$:

$$
\begin{aligned}
\mathcal{D}'_\mu \psi' &= (\partial_\mu + igb_{\mu,a}T_a - i(\partial_\mu \alpha_a)T_a + igf_{abc}\alpha_b b_{\mu,a}T_c)(1 + i\alpha_t T_t)\psi \\
&= i(\partial_\mu \alpha_t)T_t\psi + (1 + i\alpha_t T_t)(\partial_\mu \psi) + (1 + i\alpha_t T_t)(igb_{\mu,a}T_a) - gb_{\mu,a}\alpha_t(if_{atb}T_b) \\
&\quad -i(\partial_\mu \alpha_a)T_a\psi + igf_{abc}\alpha_b b_{\mu,a}T_c + \mathcal{O}(\boldsymbol{\alpha}^2) \\
&\simeq (1 + i\alpha_t T_t)(\partial_\mu + igb_{\mu,a}T_a)\psi, \qquad (8)
\end{aligned}
$$

where the commutator (4) has been used to obtain the third and fourth term on the second line.

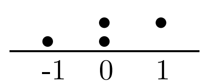
Thus, the principal equations and expectations will be invariant under the gauge transformations being discussed.

3 Group theory representations

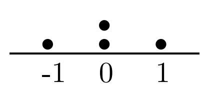
Consider the gauge group SU(2), familiar from studies of spin[2]. The generators of this group are proportional to the familiar 2×2 Pauli matrices:

$$
T_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad T_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad T_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (9)
$$

The eigenstates that result (corresponding to a spin-$\frac{1}{2}$ particle with spin "up" or "down", or $s_z = \pm\frac{1}{2}$) constitute the so-called *fundamental representation* of the group, where the number of eigenstates equals the dimension of the group, in this case, two. We can construct objects of different spin by combining multiple spin-$\frac{1}{2}$ objects into larger mulitplets. For example, we could combine two spin-$\frac{1}{2}$ objects to form a state with a z component of -1, 0 or 1, illustrated on a simple number line as follows:



In group theory notation, this combination is written $2 \otimes 2$. In reality, the middle two states (with $s_z = 0$) are not spin eigenstates, and instead we rearrange the states into a spin triplet ($s = 1$) and a spin singlet ($s = 0$):



This is written as $3 \oplus 1$.

The triplet state has three s_z eigenstates (-1, 0, +1), and can be represented by a three-component vector. The generators of SU(2) rotations for

[2]In this section, the word "spin" will be used, but this should be understood to apply equally to isospin, weak isospin, etc.

this system are 3×3 matrices, like those used for SO(3) spatial rotations:

$$J_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{10}$$

These matrices obey the same group algebra as the matrices of Equation (9), but are clearly not equivalent. In fact, these form the *adjoint representation* of the SU(2) group[3], where the number of eigenstates equals the number of generators, in this case three.

For SU(N), the adjoint representation can always be formed from an $N \otimes N$ combination. A singlet will always be produced, with a group generator equal to the identity matrix. The remaining N^2-1 matrices form the adjoint representation.

In the Standard Model, interacting fundamental matter particles (the fermions) belong to the fundamental representations of gauge groups[4]. Thus, there are two states of weak isospin (gauge group SU(2)), for example the electron and electron neutrino. Similarly, there are three colours of quark, corresponding to the dimensionality of SU(3). The gauge bosons, on the other hand, belong to the adjoint representation for each group. Thus, there are three electroweak gauge bosons (corresponding, after electroweak symmetry breaking, to the W^+, Z^0 and W^- bosons), and eight ($= 3^2 - 1$) gluons.

4 Building an interacting Lagrangian

The Dirac equation for a free fermion field ψ can be obtained from the following Lagrangian density

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi. \tag{11}$$

A gauge interaction for this fermion may be introduced by replacing ∂_μ by the covariant derivative from Equation (2):

$$\begin{aligned} \mathcal{L} &= \bar{\psi}(i\gamma^\mu \mathcal{D}_\mu - m)\psi \\ &= \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - g\bar{\psi}\gamma^\mu \boldsymbol{T} \cdot \boldsymbol{b}_\mu \psi. \end{aligned} \tag{12}$$

The final term of Equation (12) represents interactions between the fermion and the gauge field. These modify the propagation of the free

[3]They are related to the matrices with elements $(J_a)_{bc} = f_{abc}$ by $J_x = \frac{1}{\sqrt{2}} U(J_3 - J_1)U^{-1}$, $J_y = \frac{1}{\sqrt{2}} U(J_3 + J_1)U^{-1}$, $J_z = \frac{1}{\sqrt{2}} U J_2 U^{-1}$, where $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & i \\ 0 & 1-i & 0 \\ i & 0 & 1 \end{pmatrix}$.

[4]Non-interacting fermions are gauge group singlets. For example, the electron is an SU(3) singlet, and does not interact with gluon fields as a result.

fermion field, described in the first two terms. However, there is another class of terms involving just the fields $\boldsymbol{b}_\mu$. It turns out that the only Lorentz covariant object allowed by gauge symmetry that we can form from the gauge field alone is the commutator of the covariant derivative, $[\mathcal{D}_\mu, \mathcal{D}_\nu]$. This can be evaluated as follows:

$$\begin{aligned} [\mathcal{D}_\mu, \mathcal{D}_\nu] &= [\partial_\mu + ig\boldsymbol{T}\cdot\boldsymbol{b}_\mu, \partial_\nu + ig\boldsymbol{T}\cdot\boldsymbol{b}_\nu] \\ &= [\partial_\mu, \partial_\nu] + ig[\partial_\mu, \boldsymbol{T}\cdot\boldsymbol{b}_\nu] + ig[\boldsymbol{T}\cdot\boldsymbol{b}_\mu, \partial_\nu] \\ &\quad -g^2[\boldsymbol{T}\cdot\boldsymbol{b}_\mu, \boldsymbol{T}\cdot\boldsymbol{b}_\nu]. \end{aligned} \tag{13}$$

The first commutator is evidently zero. The commutator in the second term can be found by considering what happens when this operates on a wavefunction:

$$\begin{aligned} [\partial_\mu, \boldsymbol{T}\cdot\boldsymbol{b}_\nu]\psi &= \partial_\mu(\boldsymbol{T}\cdot\boldsymbol{b}_\nu\psi) - \boldsymbol{T}\cdot\boldsymbol{b}_\nu(\partial_\mu\psi) \\ &= (\partial_\mu\boldsymbol{T}\cdot\boldsymbol{b}_\nu)\psi. \end{aligned} \tag{14}$$

Note that the end result does not depend in any way on the wavefunction we temporarily introduced. Similarly, $[\boldsymbol{T}\cdot\boldsymbol{b}_\mu, \partial_\nu] = -\partial_\nu\boldsymbol{T}\cdot\boldsymbol{b}_\mu$ in the third term of Equation (13). The final term is evaluated using the group algebra, giving the following result

$$\begin{aligned} [\mathcal{D}_\mu, \mathcal{D}_\nu] &= ig\,\boldsymbol{T}\cdot(\partial_\mu\boldsymbol{b}_\nu - \partial_\nu\boldsymbol{b}_\mu) - ig^2 f_{abc} b_{\mu,a} b_{\nu,b} T_c \\ &= [ig(\partial_\mu b_{\nu,c} - \partial_\nu b_{\mu,c}) - ig^2 f_{abc} b_{\mu,a} b_{\nu,b}]T_c. \end{aligned} \tag{15}$$

In the U(1) case, the structure constants vanish, and Equation (15) is proportional to the field tensor $F_{\mu\nu} = \frac{1}{4}(\partial_\mu b_\nu - \partial_\nu b_\mu)$, familiar from electromagnetism. In the non-Abelian case, the first terms of Equation (15) also describe free fields that propagate in the vacuum much like the photon, but the final term will not vanish.

The only Lorentz scalar field propagation term that we can construct using $F_{\mu\nu} \propto [\mathcal{D}_\mu, \mathcal{D}_\nu]$ is $F_{\mu\nu}F^{\mu\nu}$. The final term of Equation (15) will yield terms proportional to $(\partial_\mu b_\nu)b^\mu b^\nu$ and $b_\mu b_\nu b^\mu b^\nu$, where for the moment the group generator indices have been suppressed. These terms ultimately correspond to interactions between the various components of the $\boldsymbol{b}_\mu$ field, something that will be explored further later in the course.

Finally, note that Equation (15) (and therefore the Lagrangian density) does not contain any terms proportional to $\boldsymbol{b}^\mu \cdot \boldsymbol{b}_\mu$. These would give the $\boldsymbol{b}_\mu$ field a mass (c.f. the Klein-Gordan equation for a masive boson), but such terms are forbidden as they break the gauge symmetry. Thus, fields produced via the symmetry principle are massless by construction.