# Tutorial 3: Working towards the Standard Model

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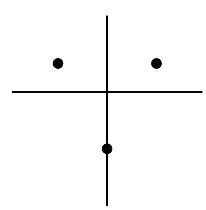
January 7, 2013

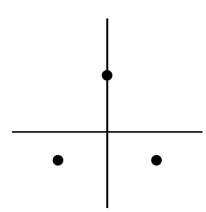
## 1 SU(3) generators and symmetry examples

In the last tutorial, we essentially completed the description of the Lagrangian terms for non-Abelian gauge fields. Although the discussion mostly referred to SU(2), the mathematics is identical for any non-Abelian group, in particular SU(3) associated with the strong nuclear force. Only the number of generators and the self-couplings described by the structure constants are different.

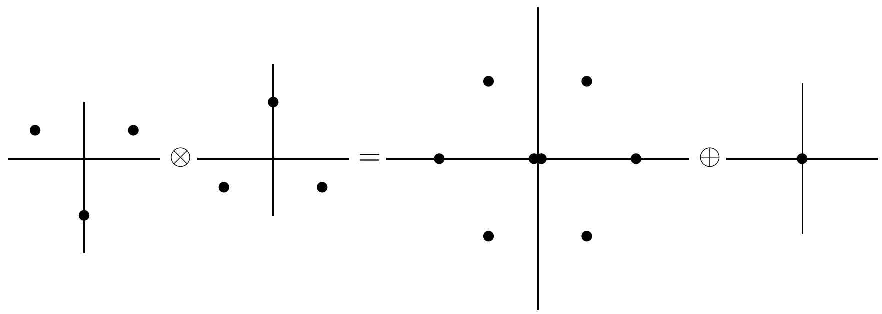
The fundamental representation of SU(3) (corresponding, e.g., to quark charges) has the same number of elements as the group's dimension, three. These can be illustrated on a two-dimensional plane, analagous to the line drawing of the SU(2) charges in the last tutorial:



In group theory notation, this is $\mathbf{3}$. In contrast to SU(2), there is also a fundamental $\bar{\mathbf{3}}$ representation, distinct from $\mathbf{3}$:



The adjoint representation can be obtained via the group product $\mathbf{3} \otimes \bar{\mathbf{3}}$. This produces a singlet state (we will see this again later) and an octet, in other words $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$. The charges of these states can be seen by imagining the $\bar{\mathbf{3}}$ charges centered on each point of the $\mathbf{3}$ graph in turn. The resulting charge diagram is as follows:



Neglecting the singlet, there are therefore 8 members of the adjoint representation, 8 SU(3) generators and 8 types of gluon. The group generators can be represented by eight $3 \times 3$ traceless Hermitian matrices[1]:

$$
\begin{aligned}
T_1 &= \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
T_4 &= \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T_5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \\
T_6 &= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T_7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T_8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned} \tag{1}
$$

These act on $3 \times 1$ column vectors of the fundamental representation, i.e. quark states. Note that there is one more independent Hermitian $3 \times 3$ matrix, proportional to the identity matrix. As this corresponds to a null operation, this belongs to the SU(3) singlet state.

These matrices have a few other interesting properties. One is that $T_1$, $T_2$ and $T_3$ together look very similar to the SU(2) generator matrices. In fact, they satisfy all the SU(2) properties and themselves form a group, acting only on the first two colours. Thus SU(2) is actually a *subgroup* of SU(3).

Finally, it should be remembered that these matrices are usually rearranged into raising and lowering operators that more elegantly describe transitions between the various states. These operators are usually denoted $I^{\pm}$ (corresponding to the SU(2) subgroup, presumably named from the analogy with isospin), $V^{\pm}$ and $U^{\pm}$, defined as follows:

$$
\begin{aligned}
I^{\pm} &= T_1 \pm iT_2, \\
V^{\pm} &= T_4 \mp iT_5, \\
U^{\pm} &= T_6 \pm iT_7.
\end{aligned} \tag{2}
$$

[1]The normalisation here is chosen such that $\mathrm{Tr}(T_a^2) = \frac{1}{2}$. The matrices used here are related to the $\lambda_a$ matrices of the lecture notes by $T_a = \frac{1}{2}\lambda_a$.

The diagonal operators $T_3$ and $T_8$ remain unchanged. Also note the relative sign change in the definition of $V^{\pm}$, this ensures that the raising operators operate in a circular fashion. More details, and the full interaction Lagrangian for strongly interacting particles, are given in the lecture notes.

### 1.1 SU(3) symmetry in baryonic systems

In the Standard Model, the SU(3) symmetry associated with the strong nuclear force is unbroken. This is in sharp contrast to the broken electroweak symmetries that will be discussed next. It also means that the structure of the group symmetry is more apparent in low-energy physics, albeit complicated significantly by the very large coupling constant associated with the strong force.

Due to the strength of this force, only colour singlets are observed in nature, meaning that we cannot directly observe isolated quarks or gluons. Relationships between the lowest-mass baryon states however strongly point to a force based on the SU(3) symmetry group, which we will review here.

The wave function for a baryon approximately factorises into four components, describing colour ($C$), spin ($S$), position ($X$) and flavour ($F$):

$$\psi = \psi_C \psi_S \psi_X \psi_F. \tag{3}$$

The constituent quarks must have half-integer spin, or else the proton and neutron could not be fermionic. Therefore, the overall wavefunction must be fully antisymmetric under exchange of any two quarks, by the spin-statistics theorem.

First, consider $\psi_X$. If we consider only ground-state baryons, all quarks will be in $s$-wave orbitals, fully symmetric under exchange of quarks.

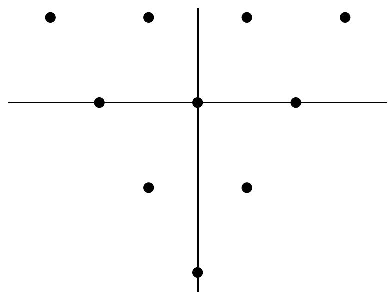
There is a nearly perfect SU(3) flavour symmetry for the lowest mass baryon (and meson) states, only broken slightly for states of different absolute strangeness. Ultimately, this comes about from the low mass of the up, down and strange quarks (all less than 100 MeV), much less than the baryon mass scale of $\sim 1$ GeV.

With this approximate symmetry in mind, it was found that the least massive baryons could be arranged into an octet of spin-$\frac{1}{2}$ particles (including the nucleons) and decuplet of spin-$\frac{3}{2}$ particles. The simplest way to obtain a decuplet of states is by the composition of three quarks, denoted by $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$.[2] Without going into details, this yields a fully symmetric decuplet, two octets with mixed exchange symmetries, and a flavour singlet:

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}. \tag{4}$$

It is the decuplet that most concerns us here. Its layout in flavour space is especially striking:

[2] The discussion applies equally to anti-baryons, where the flavour representation becomes $\bar{\mathbf{3}} \otimes \bar{\mathbf{3}} \otimes \bar{\mathbf{3}}$.



The states at the corners correspond to the obviously symmetric $uuu$, $ddd$ and $sss$ combinations, and by construction the entire multiplet is symmetric under quark exchange.

If we have three spin-$\frac{1}{2}$ quarks, then the spin wavefunction $\psi_S$ belongs to the $\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2}$ representation of SU(2). This reduces to two spin-$\frac{1}{2}$ doublets with mixed exchange symmetry and a spin-$\frac{3}{2}$ quartet. Note that spin-$\frac{3}{2}$ quarks would have a substantially more complex structure. Experimentally, the baryons in the decuplet are found to have spins of $\frac{3}{2}$, and so $\psi_S$ must correspond to this quartet. This quartet includes the states $\uparrow\uparrow\uparrow$ and $\downarrow\downarrow\downarrow$, obviously symmetric under particle exchange.

Thus, so far, the spatial, flavour and spin parts of the wavefunction for the baryon decuplet are fully symmetric under quark exchange, prompting the proposal of colour as a possible way to introduce antisymmetry into the system. If the force binding the quarks together is to be described using an SU($N$) symmetry, then the three-quark structure strongly suggests trying $N = 3$. This immediately restricts us to the singlet that results from the $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}$ representation[3]. This state is, in fact, completely *anti*symmetric with respect to particle exchange:

$$\psi_C = \frac{1}{\sqrt{6}}\left(RGB + GBR + BRG - RBG - BGR - GRB\right), \tag{5}$$

and therefore the product of all four wavefunction components is antisymmetric, as required.

## 2 A first look at the Higgs field

It was noted at the end of the second tutorial that gauge fields obtained via a symmetry principle must be massless. This describes the electromagnetic and strong forces well, but is insufficient for the weak nuclear force. The $W$ and $Z$ gauge bosons that mediate the weak force have substantial masses of 80 and 90 GeV, respectively, which has the effect of setting a short range $\sim 1/M_{W/Z}$ for weak interactions in the low energy limit.

[3]This refers now to colour, and should not to be confused with the previous discussion of SU(3) flavour symmetry.

What is perhaps less obvious is that, due to the peculiar nature of the weak interaction, all weakly interacting fermions must also be massless. It might naively be thought that a fermionic mass term $m\bar{\psi}\psi$ would always remain invariant under any transformation $\psi \to G\psi$. It turns out that this is not the case for the weak nuclear force, which acts differently on the left- and right-handed components of $\psi$. We can always rewrite $\psi$ as a sum of these components:

$$\begin{aligned}\psi &= \psi_{\mathrm{L}} + \psi_{\mathrm{R}} \\ &= \frac{1}{2}(1-\gamma^5)\psi + \frac{1}{2}(1+\gamma^5)\psi.\end{aligned} \tag{6}$$

The mass term $m\bar{\psi}\psi$ therefore has four components, as follows:

$$m\bar{\psi}\psi = m(\bar{\psi}_{\mathrm{L}}\psi_{\mathrm{L}} + \bar{\psi}_{\mathrm{L}}\psi_{\mathrm{R}} + \bar{\psi}_{\mathrm{R}}\psi_{\mathrm{L}} + \bar{\psi}_{\mathrm{R}}\psi_{\mathrm{R}}). \tag{7}$$

The matrix $\gamma^5$ is Hermitian, which allows us to evaluate $\bar{\psi}_{\mathrm{L}}$ as $\psi^\dagger(1-\gamma^5)\gamma^0 = \bar{\psi}(1+\gamma^5)$, and similarly $\bar{\psi}_{\mathrm{R}} = \bar{\psi}(1-\gamma^5)$. Recalling that $(\gamma^5)^2 = 1$, the straight terms $m\bar{\psi}_{\mathrm{L}}\psi_{\mathrm{L}}$ and $m\bar{\psi}_{\mathrm{R}}\psi_{\mathrm{R}}$ are seen to vanish, leaving just the cross terms:

$$m\bar{\psi}\psi = m(\bar{\psi}_{\mathrm{L}}\psi_{\mathrm{R}} + \bar{\psi}_{\mathrm{R}}\psi_{\mathrm{L}}). \tag{8}$$

This is clearly variant under an SU(2) symmetry operation, as $\psi_{\mathrm{L}}$ is an SU(2) doublet, while $\psi_{\mathrm{R}}$ is a singlet. Therefore, the terms in Equation (8) are not allowed in the Standard Model Lagrangian.

The Higgs field was postulated to overcome these apparent difficulties. Instead of changing the basic symmetry principles or the particle content of the Standard Model, the nature of the stable vacuum state is altered instead. In all the fields considered so far, it has been implicitly assumed that the vacuum state corresponds to $\langle\psi\rangle = 0$, up to zero-point fluctuations. This arises naturally if the potential for a particle has a minumum at zero, as in the left-hand image in Figure 1. The x-axis here corresponds to the field strength $|\psi|$, and the potential to a regular mass term $m\bar{\psi}\psi$. Adding more energy to the field[4] increases the maximum possible value for $|\psi|$.

The Higgs field $\phi$, on the other hand, has a quartic potential, illustrated on the right-hand side of Figure 1. At very high energies (the upper dotted line), this is difficult to distinguish from the quadratic case, but at low energies (lower dotted line), it is clear that the "bump" at $\psi \sim 0$ will affect the ground state significantly. In the simple one-dimensional case, there will be two (degenerate) vacuum states, each centered on one of the two minima illustrated, with the same average magnitude $\langle|\phi|\rangle = v$.

The physical Higgs field is a complex SU(2) doublet, with four real components. Despite this, the vacuum expectation value, $v$, and real fluctuations

[4] Strictly speaking, this means adding quanta to the particular mode described by $\psi(E, \boldsymbol{p})$.

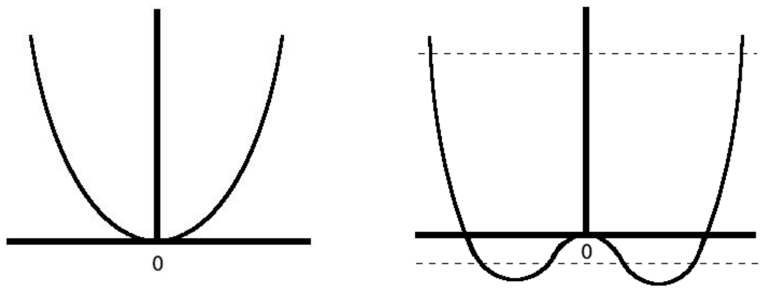



Figure 1: Left: A quadratic potential function for a massive particle. Right: The Higgs field potential, with quadratic and quartic terms.

around it, $H(x)$, can be written with full generality as

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix}, \tag{9}$$

where $v > 0$ is a real constant. This amounts to a specific choice of (spacetime-dependent) SU(2) gauge, fixing three of the four free parameters of $\phi$.

In this new vacuum, previously massless particles now appear to have mass. To see how this might work, at least for fermions, consider the Lagrangian interaction term $\lambda\phi^{\mathrm{T}}\bar{\psi}_{\mathrm{R}}\psi_{\mathrm{L}}$, where $\lambda$ is an (at this point) arbitrary coupling constant between $\psi$ and $\phi$. This, unlike Equation (8), is a gauge-invariant scalar quantity, and is thus allowed in the Lagrangian. With the specific choice of Equation (9), and supposing that we are concerned with electron-like fields $\psi_{\mathrm{L}} = \begin{pmatrix} \nu \\ e_{\mathrm{L}} \end{pmatrix}$, we have

$$\begin{aligned} \lambda\phi^{\mathrm{T}}\bar{\psi}_{\mathrm{R}}\psi_{\mathrm{L}} &= \frac{\lambda}{\sqrt{2}} \begin{pmatrix} 0 & v + H(x) \end{pmatrix} \bar{e}_{\mathrm{R}} \begin{pmatrix} \nu \\ e_{\mathrm{L}} \end{pmatrix} \\ &= \frac{\lambda}{\sqrt{2}} v\bar{e}_{\mathrm{R}}e_{\mathrm{L}} + \lambda H(x)\bar{e}_{\mathrm{R}}e_{\mathrm{L}}. \end{aligned} \tag{10}$$

Now, the first term has an identical form to the second term of Equation (8), while the second term describes an interaction between the electron field and excitations of the Higgs field $H(x)$. Adding on the Hermitian conjugate $\lambda^*\bar{\psi}_{\mathrm{L}}\psi_{\mathrm{R}}\phi$ gives a term proportional to $\bar{e}_{\mathrm{R}}e_{\mathrm{L}}$. The two together therefore give the appearance that the fermion field $e$ has mass.