Tutorial 1: Groups and symmetry

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1 What is a group?

A group is a set of objects (*elements* or *members*), which can be combined by some operation (addition, multiplication, etc.). For a description of the Standard Model (SM), it is only necessary to consider groups under multiplication, and therefore the notation of multiplication will be used for simplicity from the start. When applying this operation, four conditions must be satisfied:

1. For all elements a, b in the group, the combination ab is also a member of the group.
2. The operation must be associative, i.e. $(ab)c = a(bc)$.
3. There is an identity element e, such that $ae = ea = a$ for all elements.
4. Every element a has an inverse a^{-1}, such that $aa^{-1} = a^{-1}a = e$.

The identity element is commonly written e for generality. For multiplicative groups, e is just the number 1, or an appropriate identity matrix. Under addition, for example, the identity element is 0. From now on, "1" will replace e as labeling the identity element.

1.1 Examples

The simplest group is the trivial group:

$$\{1\}. \tag{1}$$

This has only one member, and yet satisfies all of the properties required of a group (under multiplication). We can add one member to this to construct a simple non-trivial group:

$$\{1, -1\}. \tag{2}$$

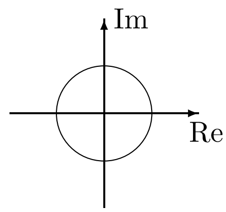
Here, each element is its own inverse.

We can further extend this, to construct a four-element complex group:

$$\{1, -1, i, -i\}. \tag{3}$$

The new elements, i and $-i$, are inverses of each other. Note that this contains $\{1, -1\}$ as a *subgroup*.

As an example of a continuous group under multiplication, take the set of complex numbers with modulus 1. This group is denoted U(1), a name which derives from the fact that this is a *unitary* group with one dimension. In the complex plane, the members of this group trace out the unit circle.



This group plays an important role in physics, as it is a part of the gauge group of the Standard Model $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$[1]. The meaning of a group product can be visualised by considering the product $\mathrm{U}(1) \times \mathbb{R}_{>0}$, where $\mathbb{R}_{>0}$ denotes the set of all positive real numbers. This yields the set of nonzero complex numbers $\mathbb{Z}_{\neq 0}$ with elements z:

$$z = re^{i\phi}, \qquad r \in \mathbb{R}_{>0}, \phi \in \mathbb{R}. \tag{4}$$

Thus, the group theoretic product of the unit circle and an semi-infinite radial axis spans the complex plane[2], including elements that are in neither group individually. Note also that U(1) and $\mathbb{R}_{>0}$ are therefore subgroups of $\mathbb{Z}_{\neq 0}$.

All of these continuous groups are of a particular form, called *Lie groups*, which we will now examine.

2 Lie groups

The defining property of a Lie group is that all elements can be reached by successive infinitesimal steps, usually starting from the identity element. Consider a member $(1 + i\epsilon)$ of U(1) a small distance ϵ from the identity. When thought of in terms of transformations, this corresponds to a small (eventually, infinitesimal) rotation of the complex plane. Rewriting ϵ as α/N, where N is a large integer, we can imagine applying this small rotation N times. Mathematically, we achieve this by multiplying $(1+i\alpha/N)$ by itself

[1] we will return to the precice meaning of SU(N) later

[2] Except zero, which has no finite multiplicative inverse. With zero included, $\mathbb{Z}$ is a group under addition.

N times, i.e. by computing $(1 + i\alpha/N)^N$. Taking the limit as $N \to \infty$, we obtain a generic member of the group:

$$\lim_{N\to\infty}\left(1 + i\frac{\alpha}{N}\right)^N = e^{i\alpha}. \tag{5}$$

This is therefore a Lie group, as only an infinitesimally small region around the identity element needs to be known in order to characterise the entire group.

Due to this property, Lie groups are often characterised in terms of their *generators*, which in our case can be thought of as unit vectors describing possible directions in which transformations can be made. The number of these directions is called the *dimension* n of the group. With a collection of generators $\boldsymbol{T}$, and associated parameters $\boldsymbol{\alpha}$ (again, one for each dimension), a generic Lie group member is written in exponential notation like Equation (5):

$$\lim_{N\to\infty}\left(1 + i\frac{\boldsymbol{\alpha}\cdot\boldsymbol{T}}{N}\right)^N = e^{i\boldsymbol{\alpha}\cdot\boldsymbol{T}}. \tag{6}$$

U(1) has only one dimension, and also only one generator. Comparing Equations (5) and (6), it is clear that the generator for U(1) is simply the number 1. When there is more than one generator, it should be remembered that the properties of groups say nothing about whether or not different members of the group will *commute*. The same observation applies to the generators of a Lie group. The commutators of the group generators define the *algebra* of the group:

$$[T_a, T_b] = T_aT_b - T_bT_a = if_{abc}T_c. \tag{7}$$

Note that the right-hand side is linear in the group generators; this is a consequence of all products of group members also being in the group.

The numbers f_{abc} are called the group's *structure constants*. If all are zero, then the group's generators (and elements) all commute; the group is *Abelian*. U(1) is an Abelian group, as all complex numbers with modulus one commute with each other. Multi-dimensional groups may be *non-Abelian*. When applied to field theories, these groups lead to self-interacting gauge fields.

2.1 A first look at SU(2)

The name SU(2) refers to the group of *special* unitary matrices with two dimensions. The term "special" means that these matrices have determinant 1, thus preserving the normalisation of state vectors upon which they act. As with any Lie group, any member of SU(2), G, can be written in terms of the group's generators $\boldsymbol{T}$

$$G = e^{i\boldsymbol{\alpha}\cdot\boldsymbol{T}}. \tag{8}$$

The generators of SU(2) are familiar, as they are proportional to the Pauli spin matrices:

$$T_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad T_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad T_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9}$$

These are Hermitian matrices with zero trace, which ensure that the group elements in Equation (8) are unitary with unit determinant. Correspondingly, $\boldsymbol{\alpha}$ is a three-component vector in the associated Hilbert space. In addition, the generator matrices have the following important properties:

$$[T_a, T_b] = i\epsilon_{abc}T_c, \text{ and } T_a^2 = \frac{1}{4}\boldsymbol{I}. \tag{10}$$

Exercise: Express $\boldsymbol{\alpha} \cdot \boldsymbol{T}$ as a 2×2 matrix. Use the Taylor series expansion of (8) to find G. Does it have the expected properties? What value of $|\boldsymbol{\alpha}|$ corresponds to a full rotation?

3 Group theory representations

To examine the different represenations of SU(2), we will use the familiar language of spin[3]. The eigenstates of a spin-$\frac{1}{2}$ particle with spin "up" or "down" ($s_z = \pm\frac{1}{2}$) constitute the so-called *fundamental representation* of the group, where the number of eigenstates equals the dimension of the group, in this case, two. We can construct objects of different spin by combining multiple spin-$\frac{1}{2}$ objects into larger mulitplets. For example, we could combine two spin-$\frac{1}{2}$ objects to form a state with a z component of -1, 0 or 1, illustrated on a simple number line as follows:

$s_{z1} = +1/2$		•	•
$s_{z1} = -1/2$	•	•	
s_z	-1	0	1

In group theory notation, this combination is written $2 \otimes 2$. In reality, the middle two states (with $s_z = 0$) are not spin eigenstates, and instead we rearrange the states into a spin triplet ($s = 1$) and a spin singlet ($s = 0$):

$s = 0$		•	
$s = 1$	•	•	•
s_z	-1	0	1

This is written as $3 \oplus 1$.

The triplet state has three s_z eigenstates (-1, 0, +1), and can be represented by a three-component vector. The generators of SU(2) rotations for

[3]In this section, the word "spin" will be used, but this should be understood to apply equally to isospin, weak isospin, etc.

this system are 3×3 matrices, like those used for SO(3) spatial rotations:

$$J_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{11}$$

These matrices obey the same group algebra as the matrices of Equation (9), but are clearly not equivalent. In fact, these form the *adjoint representation* of the SU(2) group[4], where the number of eigenstates equals the number of generators, in this case three.

For SU(N), the adjoint representation can always be formed from an $N \otimes N$ combination. A singlet will always be produced, with a group generator equal to the identity matrix. The remaining $N^2 - 1$ matrices form the adjoint representation.

In the Standard Model, interacting fundamental matter particles (the fermions) belong to the fundamental representations of gauge groups[5]. Thus, there are two states of weak isospin (gauge group SU(2)), for example the electron and electron neutrino. Similarly, there are three colours of quark, corresponding to the dimensionality of SU(3). The gauge bosons, on the other hand, belong to the adjoint representation for each group. Thus, there are three electroweak gauge bosons (corresponding, after electroweak symmetry breaking, to the W^+, Z^0 and W^- bosons), and eight ($= 3^2 - 1$) gluons.

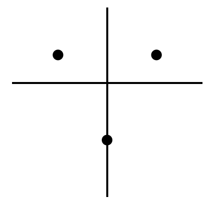
4 SU(3) generators and baryonic systems

Much of the above discussion of SU(2) applies directly to SU(3), the symmetry associated with the strong nuclear force. Only the number of generators and the self-couplings described by the structure constants are different.

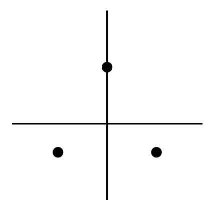
The fundamental representation of SU(3) (corresponding, e.g., to quark charges) has the same number of elements as the group's dimension, three. These can be illustrated on a two-dimensional plane, analagous to the line drawing of the SU(2) charges above:

[4] They are related to the matrices with elements $(J_a)_{bc} = f_{abc}$ by $J_x = \frac{1}{\sqrt{2}} U(J_3 - J_1)U^{-1}$, $J_y = \frac{1}{\sqrt{2}} U(J_3 + J_1)U^{-1}$, $J_z = \frac{1}{\sqrt{2}} U J_2 U^{-1}$, where $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & i \\ 0 & 1-i & 0 \\ i & 0 & 1 \end{pmatrix}$.

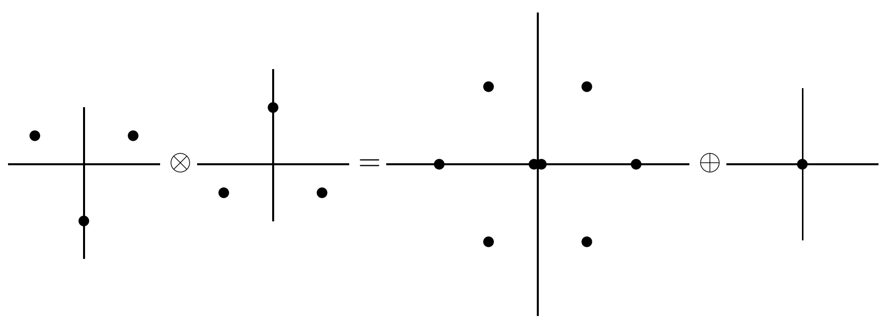
[5] Non-interacting fermions are gauge group singlets. For example, the electron is an SU(3) singlet, and does not interact with gluon fields as a result.



In group theory notation, this is $\mathbf{3}$. In contrast to SU(2), there is also a fundamental $\bar{\mathbf{3}}$ representation, distinct from $\mathbf{3}$:



The adjoint representation can be obtained via the group product $\mathbf{3} \otimes \bar{\mathbf{3}}$. This produces a singlet state (we will see this again later) and an octet, in other words $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$. The charges of these states can be seen by imagining the $\bar{\mathbf{3}}$ charges centered on each point of the $\mathbf{3}$ graph in turn. The resulting charge diagram is as follows:



Neglecting the singlet, there are therefore 8 members of the adjoint representation, 8 SU(3) generators and 8 types of gluon. The group generators can be represented by eight 3×3 traceless Hermitian matrices[6]:

$$T_1 = \frac{1}{2}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad T_2 = \frac{1}{2}\begin{pmatrix}0 & -i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad T_3 = \frac{1}{2}\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix},$$
$$T_4 = \frac{1}{2}\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}, \quad T_5 = \frac{1}{2}\begin{pmatrix}0 & 0 & -i\\ 0 & 0 & 0\\ i & 0 & 0\end{pmatrix}, \tag{12}$$
$$T_6 = \frac{1}{2}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}, \quad T_7 = \frac{1}{2}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -i\\ 0 & i & 0\end{pmatrix}, \quad T_8 = \frac{1}{2\sqrt{3}}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{pmatrix}.$$

These act on 3×1 column vectors of the fundamental representation, i.e. quark states. Note that there is one more independent Hermitian 3×3 matrix, proportional to the identity matrix. As this corresponds to a null operation, this belongs to the SU(3) singlet state.

These matrices have a few other interesting properties. One is that T_1, T_2 and T_3 together look very similar to the SU(2) generator matrices. In fact, they satisfy all the SU(2) properties and themselves form a group, acting only on the first two colours. Thus SU(2) is actually a *subgroup* of SU(3).

Exercise: Compare the pairs (T_1, T_2), (T_4, T_5) and(T_6, T_7). Are other SU(2) subgroups of SU(3) possible? Can SU(3) be written as a product of SU(2) groups? Why/why not?

Finally, it should be remembered that these matrices are usually rearranged into raising and lowering operators that more elegantly describe transitions between the various states. These operators are usually denoted $I^\pm$ (corresponding to the SU(2) subgroup, presumably named from the analogy with isospin), $V^\pm$ and $U^\pm$, defined as follows:

$$\begin{aligned} I^\pm &= T_1 \pm iT_2, \\ V^\pm &= T_4 \mp iT_5, \\ U^\pm &= T_6 \pm iT_7. \end{aligned} \tag{13}$$

The diagonal operators T_3 and T_8 remain unchanged. Also note the relative sign change in the definition of $V^\pm$, this ensures that the raising operators operate in a circular fashion. More details are given in the lecture notes.

4.1 Evidence for SU(3) colour

In the Standard Model, the SU(3) symmetry associated with the strong nuclear force is unbroken. This is in sharp contrast to the broken electroweak

[6]The normalisation here is chosen such that $\mathrm{Tr}(T_a^2) = \frac{1}{2}$. The matrices used here are related to the λ_a matrices of the lecture notes by $T_a = \frac{1}{2}\lambda_a$.

symmetries that will be discussed next. It also means that the structure of the group symmetry is more apparent in low-energy physics, albeit complicated significantly by the very large coupling constant associated with the strong force.

Due to the strength of this force, only colour singlets are observed in nature, meaning that we cannot directly observe isolated quarks or gluons. Relationships between the lowest-mass baryon states however strongly point to a force based on the SU(3) symmetry group, which we will review here.

The wave function for a baryon approximately factorises into four components, describing colour (C), spin (S), position (X) and flavour (F):

$$\psi = \psi_C \psi_S \psi_X \psi_F. \tag{14}$$

The constituent quarks must have half-integer spin, or else the proton and neutron could not be fermionic. Therefore, the overall wavefunction must be fully antisymmetric under exchange of any two quarks, by the spin-statistics theorem.

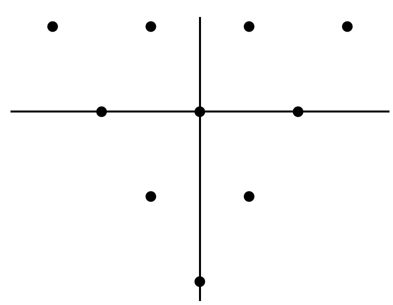
First, consider ψ_X. If we consider only ground-state baryons, all quarks will be in s-wave orbitals, fully symmetric under exchange of quarks.

There is a nearly perfect SU(3) flavour symmetry[7] for the lowest mass baryon (and meson) states, only broken slightly for states of different absolute strangeness. Ultimately, this comes about from the low mass of the up, down and strange quarks (all less than 100 MeV), much less than the baryon mass scale of ~ 1 GeV.

With this approximate symmetry in mind, it was found that the least massive baryons could be arranged into an octet of spin-$\frac{1}{2}$ particles (including the nucleons) and decuplet of spin-$\frac{3}{2}$ particles. The simplest way to obtain a decuplet of states is by the composition of three quarks, denoted by $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$.[8] Without going into details, this yields a fully symmetric decuplet, two octets with mixed exchange symmetries, and a flavour singlet:

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}. \tag{15}$$

It is the decuplet that most concerns us here. Its layout in flavour space is especially striking:



[7] Not to be confused with the SU(3) *colour* symmetry.

[8] The discussion applies equally to anti-baryons, where the flavour representation becomes $\bar{\mathbf{3}} \otimes \bar{\mathbf{3}} \otimes \bar{\mathbf{3}}$.

The states at the corners correspond to the obviously symmetric uuu, ddd and sss combinations, and by construction the entire multiplet is flavour symmetric under quark exchange.

If we have three spin-$\frac{1}{2}$ quarks, then the spin wavefunction ψ_S belongs to the $\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2}$ representation of SU(2). This reduces to two spin-$\frac{1}{2}$ doublets with mixed exchange symmetry and a spin-$\frac{3}{2}$ quartet. Note that spin-$\frac{3}{2}$ quarks would have a substantially more complex structure. Experimentally, the baryons in the decuplet are found to have spins of $\frac{3}{2}$, and so ψ_S must correspond to this quartet. This quartet includes the states $\uparrow\uparrow\uparrow$ and $\downarrow\downarrow\downarrow$, obviously symmetric under particle exchange.

Thus, so far, the spatial, flavour and spin parts of the wavefunction for the baryon decuplet are fully symmetric under quark exchange, prompting the proposal of colour as a possible way to introduce antisymmetry into the system. If the force binding the quarks together is to be described using an SU(N) symmetry, then the three-quark structure strongly suggests trying $N = 3$. This immediately restricts us to the singlet that results from the $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$ representation. This state is, in fact, completely *anti*symmetric with respect to particle exchange:

$$\psi_C = \frac{1}{\sqrt{6}} \left(RGB + GBR + BRG - RBG - BGR - GRB \right), \qquad (16)$$

and therefore the product of all four wavefunction components is antisymmetric, as required.