Tutorial 3: The Higgs mechanism and mass mixings

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1 The Standard Model Lagrangian

We are now ready to write down the complete Lagrangian density for the Standard Model. The gauge symmetry groups are:

- A $U(1)_Y$ symmetry, acting on hypercharge, with associated field B^μ.
- An $SU(2)_L$ symmetry acting on left-handed fermions and the Higgs field, with associated field $\boldsymbol{W}^\mu$.
- An $SU(3)_C$ symmetry acting on quarks, with associated field $\boldsymbol{G}^\mu$.

The matter fields are the following (for one generation - the other two generations are identically structured):

- A left-handed lepton SU(2) doublet, $\ell_\text{L} = \begin{pmatrix} \nu_\text{L} \\ e_\text{L} \end{pmatrix}$.
- A right-handed electron SU(2) singlet, e_R, and theoretically, a possible right-handed neutrino field ν_R[1].
- A left-handed quark SU(2) doublet, $q_\text{L} = \begin{pmatrix} u_\text{L} \\ d_\text{L} \end{pmatrix}$.
- Two right-handed quark SU(2) singlets, u_R and d_R.

Finally, there is the Higgs field ϕ, an SU(2) doublet. The hypercharges and gauge group representations of all these fields are summarised in Table 1.

[1] As it does not participate in any SM interaction, the ν_R is undetectable, and its only role would be to give the neutrino fields mass. As neutrinos were thought for a long time to be massless, and there are still other possible ways for neutrinos to gain mass, it is usually not considered to be part of the Standard Model. Here we include it just as an exercise.

Particle	Y_f	SU(2)-plet	SU(3)-plet
ℓ_L	-1	**2**	**1**
$(\nu_\mathrm{R}$	0	**1**	**1**)
e_R	-2	**1**	**1**
q_L	$\frac{1}{3}$	**2**	**3**
u_R	$\frac{4}{3}$	**1**	**3**
d_R	$-\frac{2}{3}$	**1**	**3**
ϕ	1	**2**	**1**
B	0	**1**	**1**
$\boldsymbol{W}$	0	**3**	**1**
$\boldsymbol{G}$	0	**1**	**8**

Table 1: Hypercharge values and gauge representations for the Standard Model fields.

Using these fields, the SM Lagrangian *for one fermion generation* may be written as follows:

$$\mathcal{L} = \sum_f \bar{\psi}_f i\gamma^\mu \mathcal{D}_\mu^f \psi_f \tag{1a}$$

$$- \frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}\boldsymbol{W}_{\mu\nu}\cdot\boldsymbol{W}^{\mu\nu} - \frac{1}{4}\boldsymbol{G}_{\mu\nu}\cdot\boldsymbol{G}^{\mu\nu} \tag{1b}$$

$$+ (\mathcal{D}_\mu^\phi \phi)^\dagger \mathcal{D}^{\phi\mu}\phi - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 \tag{1c}$$

$$- \frac{\sqrt{2}}{v}\big[\bar{\ell}_\mathrm{L}\phi Y_e e_\mathrm{R} + \text{h.c.} \tag{1d}$$

$$+ \begin{pmatrix} -\bar{e}_\mathrm{L} & \bar{\nu}_\mathrm{L} \end{pmatrix} \phi^* Y_\nu \nu_\mathrm{R} + \text{h.c.} \tag{1e}$$

$$+ \bar{q}_\mathrm{L}\phi Y_d d_\mathrm{R} + \text{h.c.} \tag{1f}$$

$$+ \begin{pmatrix} -\bar{d}_\mathrm{L} & \bar{u}_\mathrm{L} \end{pmatrix} \phi^* Y_u u_\mathrm{R} + \text{h.c.}\big] \tag{1g}$$

$$+ \theta\frac{\alpha_s}{8\pi}\epsilon^{\mu\nu\rho\sigma}\boldsymbol{G}_{\mu\nu}\cdot\boldsymbol{G}_{\rho\sigma}\,. \tag{1h}$$

Let us examine these lines one by one.

The first line, (1a), contains kinetic and gauge interaction terms for all the fermion fields, indicated by the sum over f. The covariant derivative is now also labeled by f, as its form changes depending on the properties of the associated fermions. The most complex covariant derivative is the one for the left-handed quarks, which has the following form:

$$\begin{aligned}\mathcal{D}_\mu^{q_\mathrm{L}} &= \partial_\mu + i\frac{g'}{2}Y_{q_\mathrm{L}}B_\mu + ig\boldsymbol{T}\cdot\boldsymbol{W}_\mu + ig_s\boldsymbol{T_s}\cdot\boldsymbol{G}_\mu \\ &= \partial_\mu + i\frac{g'}{2}Y_{q_\mathrm{L}}B_\mu + i\frac{g}{2}\boldsymbol{\tau}\cdot\boldsymbol{W}_\mu + i\frac{g_s}{2}\boldsymbol{\lambda}\cdot\boldsymbol{G}_\mu.\end{aligned} \tag{2}$$

This has been expressed using two different conventions for the group generator normalisation. The first, used in these notes so far, has $\mathrm{Tr}(T_a^2) = \frac{1}{2}$ and is more convenient for whole matrix operations. The second is more convenient when calculating with elements of the generator matrices, as they tend to be whole integers. The factor of $\frac{1}{2}$ in the U(1) term $i\frac{g'}{2}Y_{q_\mathrm{L}}B_\mu$ is purely conventional, but related to the above usage.

In (1b), we have the kinetic terms for the gauge bosons. The field tensors are related to the commutator of the relevant covariant derivative, as shown in the second tutorial. For example, $B_{\mu\nu} = -\frac{i}{g'}[\mathcal{D}_\mu, \mathcal{D}_\nu] = \partial_\mu B_\nu - \partial_\nu B_\mu$.

Lines (1c) to (1g) contain terms involving the Higgs field ϕ. The potential terms on line (1c) allow the vacuum expectation value $\langle\phi\rangle$ to be non-zero as discussed in the previous tutorial. The remaining lines allow the fermion fields to acquire mass once this is done, parameterised by the Yukawa coupling constants Y_e, Y_ν, Y_u and Y_d. Again, the neutrino terms in (1e) are shown only for illustration, see the footnote on page 1. As we are only considering one generation for now, the Yukawa constants are simply numbers, equal in magnitude to the appropriate particle mass. The mass terms are not themselves Hermitian, and so their Hermitian conjugates ("h.c.") must also be added to the Lagrangian so that the whole remains Hermitian. For example, the "h.c." term in (1d), giving mass to the electron field, is $\bar{e}_\mathrm{R} Y_e^* \phi^\dagger \ell_\mathrm{L}$.

Exercise: Why are the left-handed fields reversed in (1e) and (1g)?
Hint: What terms would result if they were replaced with $\bar{\ell}_\mathrm{L}$ and $\bar{d}_\mathrm{L}$? Why are these terms forbidden?

Finally, we have the term in (1h). This is the so-called "QCD θ" term, and is in principle allowed by all of the symmetries of the SM. However, the presence of the fully antisymmetric tensor $\epsilon^{\mu\nu\rho\sigma}$ means that this term violates CP, an effect which is not observed in strong interactions. Experimental evidence suggests that the parameter θ is less than about 10^{-9}, although there is no clear reason as to why it should be so small.

2 Electroweak symmetry breaking and the boson sector

In the stable vacuum at low energy, the Higgs field has a non-zero vacuum expectation value. In actual fact, an infinite number of equivalent vacua exist, which in turn has important consequences for the B^μ and $\boldsymbol{W}^\mu$ fields.

The potential, taken from Equation (1c), may be rewritten (up to a constant term which we may neglect) in terms of two new parameters v and m_h:

$$\begin{aligned} V(\phi) &= \mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2 \\ &= \frac{m_h^2}{2v^2}\left(\phi^\dagger\phi - \frac{v^2}{2}\right)^2 + \text{const.} \end{aligned} \tag{3}$$

In the second form, there is clearly a ring of minima, all satisfying $\phi^\dagger\phi = \frac{v^2}{2}$, or $|\phi| = v/\sqrt{2}$. The values of μ and λ in terms of m_h and v are easily found

$$\begin{aligned} \lambda &= \frac{m_h^2}{2v^2} \\ \mu^2 &= \frac{m_h^2}{2v^2} \cdot 2 \cdot \left(-\frac{v^2}{2}\right) = -\frac{1}{2}m_h^2. \end{aligned} \tag{4}$$

Now we shall consider the effect of a small perturbation from some point with $|\phi| = v/\sqrt{2}$. For the moment, we shall neglect the fact that ϕ is an SU(2) doublet, and just assume it is a complex field. Without loss of generality within this assumption, we can consider starting from the point $\phi = v/\sqrt{2}$. We may write the perturbation in terms of two real parameters, ψ and χ:

$$\phi = \frac{1}{\sqrt{2}}\left(v + \chi(x) + i\psi(x)\right). \tag{5}$$

Now we place this expression into the Higgs kinetic and self-interaction part of the Standard Model Lagrangian, for now neglecting gauge interactions, so that $\mathcal{D}_\mu^\phi = \partial_\mu$.

$$\begin{aligned} \mathcal{L}_{\text{Higgs}} &= (\partial_\mu\phi)^\dagger\partial^\mu\phi - V(\phi) \\ &= \frac{1}{2}\partial_\mu(\chi - i\psi)\partial^\mu(\chi + i\psi) - \frac{m_h^2}{2v^2}\cdot\frac{1}{4}\left\{(v+\chi-i\psi)(v+\chi+i\psi) - v^2\right\}^2 \\ &= \frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{1}{2}\partial_\mu\psi\partial^\mu\psi - \frac{m_h^2}{8v^2}\left\{2v\chi + \chi^2 + \psi^2\right\}^2 \\ &= \frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{1}{2}\partial_\mu\psi\partial^\mu\psi - \frac{m_h^2}{8v^2}\left[4v^2\chi^2 + 4v\chi(\chi^2+\psi^2) + (\chi^2+\psi^2)^2\right] \\ &= \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{1}{2}m_h^2\chi^2 + \frac{1}{2}\partial_\mu\psi\partial^\mu\psi - \mathcal{O}(\chi^3, \chi\psi^2). \end{aligned} \tag{6}$$

Thus, neglecting third- and higher-order terms, we see that the field χ has gained a mass term $-\frac{1}{2}m_h^2\chi^2$. Substituting into the Euler-Lagrange equation, we find that χ obeys the Klein-Gordon equation for a particle of mass m_h, as anticipated by the variable name:

$$\begin{aligned} \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\chi)} - \frac{\partial\mathcal{L}}{\partial\chi} = 0 \Rightarrow \partial_\mu\left(\frac{1}{2}\cdot 2\partial^\mu\chi\right) - \left(-\frac{1}{2}m_h^2\cdot 2\chi\right) &= 0 \\ \text{or}\quad \partial_\mu\partial^\mu\chi + m_h^2\chi &= 0. \end{aligned} \tag{7}$$

In the Standard Model, this field represents an observable Higgs boson.

What about ψ? This has no mass term, and corresponds to rotations around the bottom of the Higgs potential. It is called a *Goldstone boson*, and its appearance is a general feature of spontaneous symmetry breaking. It is, however, not directly observable, thanks to gauge symmetry. In this simple example, the ψ part of the transformation Equation (5) can be removed by a local infinitesimal U(1) rotation $G = e^{-i\psi(x)/\sqrt{2}}$. This amounts to making a specific choice of gauge. After making this choice, the underlying U(1) symmetry is no longer apparent; it has been *broken*.

The extension to an SU(2) Higgs doublet is similar, but more algebraically challenging. Three Goldstone bosons emerge, which can be removed by fixing the $\mathrm{U}(1)_Y \times \mathrm{SU}(2)_L$ gauge, breaking the symmetry of the Lagrangian. The combined symmetry group has four parameters, only three of which are needed to remove the Goldstone bosons. The fourth parameter is associated with the electromagnetic U(1) symmetry that remains.

2.1 Electroweak boson mass

Now we consider the boson fields after $\mathrm{U}(1)_Y \times \mathrm{SU}(2)_L$ symmetry breaking occurs. All the essential features are in the covariant derivative $\mathcal{D}^\phi_\mu$. Written explicitly as a 2×2 matrix in $\mathrm{SU}(2)_L$ space, we find that

$$\begin{aligned} \mathcal{D}^\phi_\mu &= \partial_\mu + i\frac{g'}{2} Y_\phi B_\mu + i\frac{g}{2} \boldsymbol{\tau} \cdot \boldsymbol{W}_\mu \\ &= \begin{pmatrix} \partial_\mu + i\frac{g'}{2} B_\mu + i\frac{g}{2} W^0_\mu & i\frac{g}{2}\left(W^1_\mu - iW^2_\mu\right) \\ i\frac{g}{2}\left(W^1_\mu + iW^2_\mu\right) & \partial_\mu + i\frac{g'}{2} B_\mu - i\frac{g}{2} W^0_\mu \end{pmatrix} \end{aligned} \quad (8)$$

recalling that $Y_\phi = 1$.

We begin with the usual definition of SU(2) raising and lowering operators, otherwise known as the charged W boson fields:

$$W^\pm_\mu = \frac{1}{\sqrt{2}}\left(W^1_\mu \mp iW^2_\mu\right). \quad (9)$$

We then have B_μ and W^0_μ remaining along the leading diagonal, describing neutral-current electroweak interactions. These fields have the same quantum numbers and may mix. With a choice of $\langle\phi\rangle = \begin{pmatrix} 0 \\ v \end{pmatrix}$, only the combination in the lower right corner of the matrix in Equation (8) acquires mass. With appropriate normalisation, we define this field to be Z_μ

$$Z_\mu = \frac{gW^0_\mu - g'B_\mu}{\sqrt{g^2 + g'^2}} = \cos\theta_W W^0_\mu - \sin\theta_W B_\mu, \quad (10)$$

where $\tan\theta_W = g'/g$ defines the Weinberg angle θ_W. The orthogonal field is denoted A_μ:

$$A_\mu = \frac{g'W^0_\mu + gB_\mu}{\sqrt{g^2 + g'^2}} = \sin\theta_W W^0_\mu + \cos\theta_W B_\mu. \quad (11)$$

Substituting these into Equation (8), we obtain the following, after simplification:

$$\mathcal{D}^{\phi}_{\mu}\phi = \begin{pmatrix} \partial_\mu + ig'\cos\theta_W A_\mu + i\frac{g\cos 2\theta_W}{2\cos\theta_W}Z_\mu & i\frac{g}{\sqrt{2}}W^+_\mu \\ i\frac{g}{\sqrt{2}}W^-_\mu & \partial_\mu - i\frac{g}{2\cos\theta_W}Z_\mu \end{pmatrix} \begin{pmatrix} 0 \\ \frac{v+H(x)}{\sqrt{2}} \end{pmatrix}$$
$$= \begin{pmatrix} i\frac{g}{2}\left(W^+_\mu\right)(v+H(x)) \\ \frac{1}{\sqrt{2}}\left(\partial_\mu - i\frac{g}{2\cos\theta_W}Z_\mu\right)(v+H(x)) \end{pmatrix}. \tag{12}$$

The modulus of the coefficient of A_μ in the top-left element of the $\mathcal{D}^{\phi}_{\mu}$ matrix, $g'\cos\theta_W$, is identified with the unit of electric charge, e.

The interactions between the gauge fields and the Higgs field are described by the modulus square of Equation (12), $(\mathcal{D}^{\phi}_{\mu}\phi)^\dagger\mathcal{D}^{\phi\mu}\phi$. Observing that $W^{-\dagger}_\mu = W^+_\mu$ and that Z_μ is Hermitian, this results in terms proportional to $v^2W^+_\mu W^{\mu-}$ and $v^2Z_\mu Z^\mu$. These have the usual quadratic form for boson mass terms. Like the fermions, the gauge bosons acquire mass from their interactions with the Higgs field. Unlike the fermions, their masses are fixed once v, g and g' are known. The remaining terms arising from $(\mathcal{D}^{\phi}_{\mu}\phi)^\dagger\mathcal{D}^{\phi\mu}\phi$ describe the motion of the Higgs field H and its interactions with the weak boson fields. No terms involving A_μ appear, and therefore this field remains massless. It also does not interact with H.

2.2 Coupling of the Z and photon to fermions

Let's take another look at the diagonal terms of the covariant derivative in Equation (8). Generalising to other SU(2) doublets, we can without loss of generality write the interaction part as[2]

$$i\frac{g'}{2}Y_f B_\mu + igI_3W^0_\mu, \tag{13}$$

where I_3 is the third component of weak isospin for the field f (*i.e.* $\pm\frac{1}{2}$ for the two members of a weak doublet, or zero for a right-handed singlet). Inverting Equations 10 and 11, we can express this in terms of the physical boson fields (dropping the common factor i for clarity):

$$\frac{g'}{2}Y_f B_\mu + gI_3W^0_\mu = \begin{pmatrix} \frac{g'}{2}Y_f & gI_3 \end{pmatrix} \begin{pmatrix} B_\mu \\ W^0_\mu \end{pmatrix} \tag{14}$$
$$= \begin{pmatrix} \frac{g'}{2}Y_f & gI_3 \end{pmatrix} \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix}$$
$$= \begin{pmatrix} \frac{g'}{2}Y_f\cos\theta_W + gI_3\sin\theta_W & gI_3\cos\theta_W - \frac{g'}{2}Y_f\sin\theta_W \end{pmatrix} \begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix}.$$

[2]The quarks have strong interaction terms, but these are unaffected by what follows.

Recalling that $g \sin\theta_W = g' \cos\theta_W = e$, the factor multiplying A_μ simplifies to

$$\frac{g'}{2} Y_f \cos\theta_W + g I_3 \sin\theta_W = e\left(\frac{Y_f}{2} + I_3\right), \tag{15}$$

and therefore we define the electric charge of the field f to be equal to $Q_f = \frac{Y_f}{2} + I_3$. In fact, the hypercharge values in Table 1 are assigned based on this relationship. The coupling to the Z boson is more complicated. Expressing Y_f in terms of the weak isospin and electric charge and g' in terms of g and θ_W, the factor multiplying Z_μ in Equation (14) becomes

$$\begin{aligned} g I_3 \cos\theta_W - \frac{g'}{2} Y_f \sin\theta_W &= g I_3 \cos\theta_W - g(Q_f - I_3)\tan\theta_W \sin\theta_W \\ &= \frac{g}{\cos\theta_W}\left(I_3 \cos^2\theta_W - (Q_f - I_3)\sin^2\theta_W\right) \\ &= \frac{g}{\cos\theta_W}\left(I_3 - Q_f \sin^2\theta_W\right). \end{aligned} \tag{16}$$

The factor $\frac{g}{\cos\theta_W}$ is by convention absorbed into the coupling strength, and thus the weak "charge" for coupling to the Z boson is $(I_3 - Q_f \sin^2\theta_W)$.

3 Lepton masses: the CKM and PMNS matrices

We end this section of the course on gauge symmetry with a few notes about one of the most important effects of extending the SM to three generations. The first place in the Lagrangian that the fermion fields enter is in line (1a), where the covariant derivatives act on them. When more than one generation exists, we can choose to align the generations such that the simple sum over fermions remains valid, producing terms like $\bar{u}_\mathrm{L} i\gamma^\mu \mathcal{D}_\mu^{q_\mathrm{L}} u_\mathrm{L}$ but not $\bar{c}_\mathrm{L} i\gamma^\mu \mathcal{D}_\mu^{q_\mathrm{L}} u_\mathrm{L}$, for example. These fields (u_L, c_L etc.) define the *interaction basis* of the fermion fields, and each generation remains indepdendent.

However, by making this choice, we are then not able to prevent additional Higgs Yukawa terms such as $\begin{pmatrix}\bar{c}_\mathrm{L} & \bar{s}_\mathrm{L}\end{pmatrix} \phi Y_{u21} d_\mathrm{R}$. This mixes the first and second generations, but satisfies all SM gauge symmetries because the SU(2) doublet $\begin{pmatrix}\bar{c}_\mathrm{L} & \bar{s}_\mathrm{L}\end{pmatrix}$ transforms in exactly the same way as $\begin{pmatrix}\bar{u}_\mathrm{L} & \bar{d}_\mathrm{L}\end{pmatrix}$.[3] After all flavour combinations are taken into account, and considering just quarks for the moment, this means that Y_u and Y_d now become 3×3 (complex) matrices, rather than just numbers.

These matrices can be diagonalised, as shown in the lecture notes, by rotating the quark fields in flavour space with a unitary transformation ($u'_\mathrm{L} = U_u^\dagger u_\mathrm{L}$ etc.). This defines the *mass basis* of the flavours, so-called because it diagonalises the mass terms. It describes the states that would propagate freely in the absence of interactions.

[3] So too does $\begin{pmatrix}\bar{t}_\mathrm{L} & \bar{b}_\mathrm{L}\end{pmatrix}$.

Now we consider the gauge interactions between mass eigenstates. All of the terms in Equation (1a) involving right-handed fields are of the form $\bar{u}_\mathrm{R} X u_\mathrm{R}$ etc., where X is some operator, and are in fact unaffected by this transformation. Similarly, the electromagnetic and strong interaction terms for left-handed quarks are of the form $\bar{q}_\mathrm{L} X q_\mathrm{L}$, where X is proportional to the SU(2) identity matrix. These terms are also unaffected by the transformation, so that these forces remain flavour-diagonal.

It is a different matter when we come to weak interactions, in particular charged current interactions. The Lagrangian term is again of the form $\bar{q}_\mathrm{L} X q_\mathrm{L}$, but X has off-diagonal terms equal to those shown in Equations (8) and (12). When expanded, this results in terms like $\bar{u}_\mathrm{L} \gamma^\mu W_\mu^- d_\mathrm{L}$, which transform to $\bar{u}'_\mathrm{L} U_u^\dagger \gamma^\mu W_\mu^- U_d d'_\mathrm{L}$. The matrix $U_u^\dagger U_d$ is called the *CKM* matrix, after Cabbibo, Kobayashi and Maskawa, and it describes the mixings between mass eigenstates that can occur in charged-current weak interactions.

If we assume that the neutrino mass terms in Equation (1e) exist, then a similar matrix can be obtained for the leptons. This is the so-called PMNS matrix (after Pontecorvo, Maki, Nakagawa and Sakata), and could be responsible for neutrino oscillations. In this case, the form of the matrix is similar to the CKM matrix, although the numerical values of the elements are very different. However, the ν_R is essentially unobservable, and other possibilities for neutrinos gaining mass exist, yielding different kinds of lepton flavour mixing. We will return to this topic later in the course.