Tutorial 5: The parton model of hadrons

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1 Fermion-fermion scattering: recap

In this tutorial, we will discuss the nature of hadrons, and some of the experimental evidence for partons (quarks and gluons) inside the proton and neutron. These experiments usually focus on interactions of composite hadrons (or nuclei) with elementary leptons, in so-called *deep inelastic scattering*, or DIS. For lack of time, we will only consider electromagnetic interactions, however the same methods have been used to probe the charged- and neutral-current weak interactions of nuclei.

We begin by considering interactions between elementary fermions, before discussing the parton model proper. We finished the last tutorial with the differential cross section for elastic t-channel EM scattering of two dissimilar fermions in the centre of momentum (c.m.) frame. The diagram for this process is redrawn in Figure 1 using symbols conventionally used when describing DIS. The cross section in the c.m. frame (denoted by hats) is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\hat{\Omega}} = \frac{q_1^2 q_2^2}{32\pi^2 s}\frac{s^2+u^2}{t^2}. \tag{1}$$

The right hand side of the equation is fully Lorentz invariant, however the left side is not. We note that in the c.m. frame, the Mandelstam variable $t = q^2$ may be written as

$$\begin{aligned} t &= (\hat{k} - \hat{k}')^2 \\ &= -2\hat{E}^2(1-\cos\hat{\theta}) \\ &= -\frac{s}{2}(1-\cos\hat{\theta}) \end{aligned} \tag{2}$$

if the angle of deflection is $\hat{\theta}$ and $s = (\hat{k}+\hat{p})^2 = 4\hat{E}^2$.

Exercise: Draw a diagram of the c.m. frame momenta and fill in the missing steps of Equation (2).

Therefore, the differential change in $-t$ as $\hat{\theta}$ varies with s constant is

$$\mathrm{d}(-t) = -\frac{s}{2}\,\mathrm{d}(\cos\hat{\theta}) = \frac{s}{4\pi}\int_\phi \mathrm{d}\hat{\Omega}. \tag{3}$$

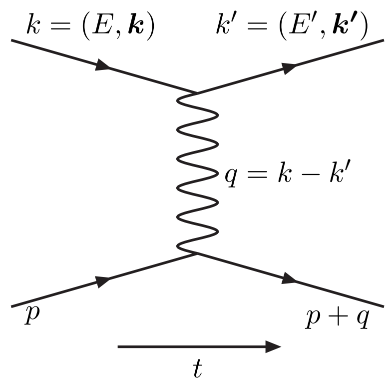

Figure 1: Diagram of first-order electromagnetic scattering of two non-identical fermions.

We can use this to rewrite Equation (1) in a fully Lorentz invariant form:

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}(-t)} &= \frac{4\pi}{s} \cdot \frac{q_1^2 q_2^2}{32\pi^2 s} \frac{s^2 + u^2}{t^2} \\ &= \frac{q_1^2 q_2^2}{8\pi t^2} \frac{s^2 + u^2}{s^2}. \end{aligned} \tag{4}$$

When considering DIS, Equation (4) is usually rewritten in terms of these new variables:

$$\begin{aligned}
&\text{C.M. energy} && s = 2k \cdot p \\
&\text{Momentum transfer} && Q^2 = -q^2 = 2k \cdot k' = -t \\
&\text{Inelasticity} && y = \frac{2p \cdot q}{s} = 1 + \frac{u}{s} \\
&\text{Björken scaling variable} && x = \frac{k \cdot k'}{k \cdot p} = \frac{Q^2}{sy}
\end{aligned} \tag{5}$$

Exercise: At most, how many of the four DIS variables are independent?

Exercise: If the fermions in Figure 1 are fundamental, $p^2 = (p+q)^2$. What does this imply for x?

In what follows, we will also normalise the charges to e, writing $q_i^2 = Q_i^2 e^2 = 4\pi Q_i^2 \alpha$. Expressed using these variables, Equation (4) becomes

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2} &= Q_1^2 Q_2^2 \frac{2\pi\alpha^2}{Q^4} \left[1 + (1-y)^2\right] \\ &= Q_1^2 Q_2^2 \frac{2\pi\alpha^2}{Q^4} \left[y^2 + 2(1-y)\right] \end{aligned} \tag{6}$$

2 Deep inelastic scattering

Next, we consider inelastic scattering of a fermion (e.g. an electron) from a hadron (e.g. a proton). Hadrons are composite particles, and therefore the interaction does not have the same simple form as Equation (6). Recall that the matrix element for Figure 1 could be written as the product of two fermion tensors, evaluated independently for each interacting particle:

$$|M_{fi}|^2 = \frac{q_1^2 q_2^2}{Q^2} L_1^{\mu\nu} L_{2\mu\nu}. \tag{7}$$

For the hadron case, we simply replace one fermion tensor with an unknown hadron tensor $W_{\mu\nu}$:

$$|M_{fi}|^2 = \frac{q_1^2 q_2^2}{Q^2} L^{\mu\nu} W_{\mu\nu}. \tag{8}$$

The hadron tensor is constrained by the same symmetries as the lepton tensor, but with arbitrary non-perturbative coefficients. The fact that the collision is inelastic also means that the relationship $Q^2 = sy$ no longer holds, and the interaction cross section now depends on the Björken scaling variable x. The differential cross section is therefore parameterised in terms of arbitrary *form factors* $F_i(x, Q^2)$, so that Equation (6) becomes

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x\,\mathrm{d}Q^2} = Q_1^2 Q_2^2 \frac{4\pi\alpha^2}{xQ^4} \left[y^2 x F_1(x, Q^2) + (1-y) F_2(x, Q^2) \right]. \tag{9}$$

Note that some of the coefficient factors have changed with respect to Equation (6), this is purely conventional. Also note that when weak interactions are considered, other terms may arise, not shown here.

The form factors for a hadron must be determined experimentally, however the quark parton model makes specific predictions about their properties. We consider the hadron to be composed of multiple *partons*, only one of which scatters with the incoming fermion. This is illustrated in Figure 2. It is assumed that this parton carries a fraction ξ of the hadron's four-momentum.[1] We assume that the partons are themselves fermionic, and that the scattering is elastic at the parton level. This latter assumption means that we can relate ξ to the DIS variables in Equation (5) by requiring that the parton's mass is unchanged by the collision. Therefore

$$\begin{aligned} (\xi p)^2 &= (\xi p + q)^2 \\ \Rightarrow 0 &= 2\xi p \cdot q + q^2 \\ \Rightarrow \xi &= \frac{-q^2}{2p \cdot q} = \frac{Q^2}{sy} = x. \end{aligned} \tag{10}$$

[1] The parton's momentum perpendicular to the hadron's motion is zero; strictly speaking, this is valid only in the so-called *infinite momentum frame*, where any transverse momentum may be neglected relative to p.

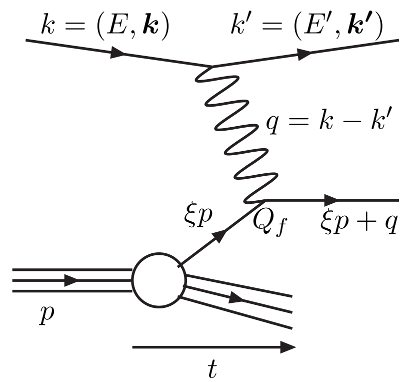

Figure 2: Diagram of deep inelastic scattering of a hadron.

Thus, the Björken variable x is identified as the momentum fraction carried by the interacting parton.

Any hadron will contain a variety of different parton flavours, which we must sum over to find the total form factor. The probability of finding a parton a with a momentum xp in hadron A is described by a *parton density function* (pdf) $f_A^a(x)$. Therefore, we can write the observed double-differential cross section as

$$\begin{aligned}\frac{\mathrm{d}\sigma(x, Q^2)}{\mathrm{d}x\,\mathrm{d}Q^2} &= \sum_a f_A^a(x) \frac{\mathrm{d}\sigma_a(Q^2)}{\mathrm{d}Q^2} \\ &= Q_1^2 \frac{2\pi\alpha^2}{xQ^4}\left[y^2 + 2(1-y)\right] x \sum_a Q_a^2 f_A^a(x). \qquad (11)\end{aligned}$$

Comparing with Equation (6), we deduce the following:

$$\begin{aligned}F_1(x, Q^2) &= \frac{1}{2}\sum_a Q_a^2 f_A^a(x) \\ F_2(x, Q^2) &= x \sum_a Q_a^2 f_A^a(x), \qquad (12)\end{aligned}$$

which leads to two important observations:

Björken scaling: F_1 and F_2 depend only on x, and not on the energy scale Q^2. This contrasts strongly with, say, resonance production, where the cross section varies strongly with Q^2.

The Callan-Gross relation: $F_2(x) = 2xF_1(x)$, sensitive to the parton spin.

Both of these were confirmed at SLAC, as shown in Figure 3. These, and other observations, led to the adoption of the quark parton model.

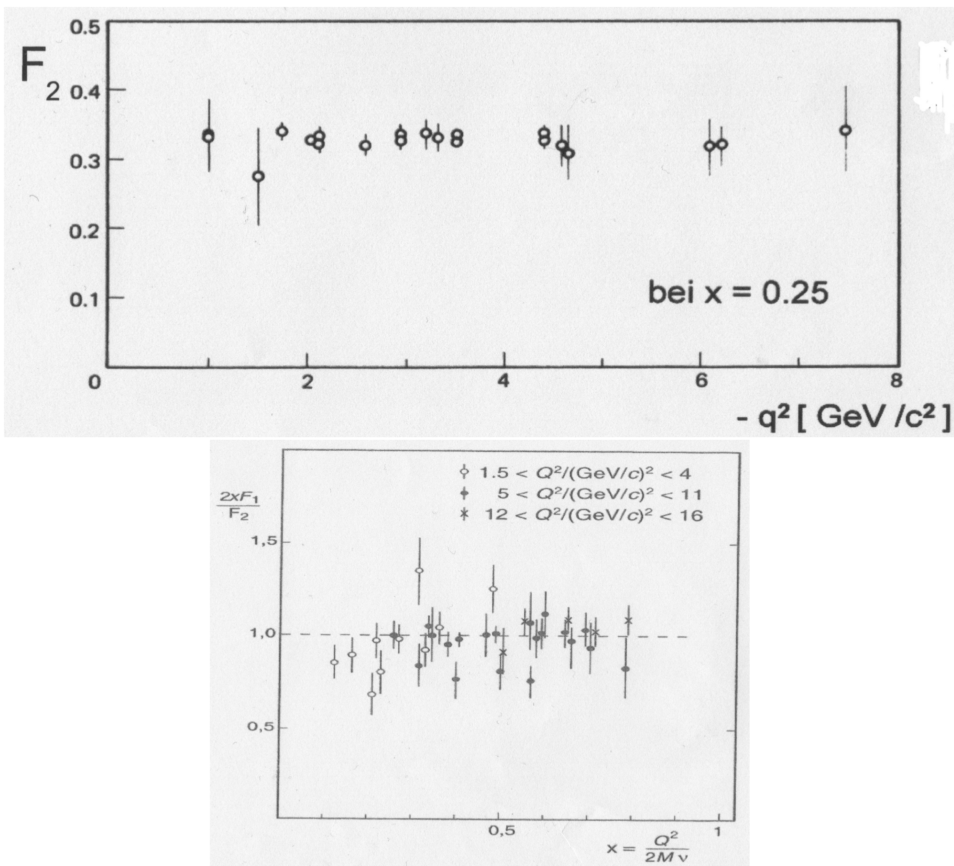

Figure 3: Top: Illustration of Björken scaling, where $F_2(x = 0.25)$ is constant over a wide range of Q^2 values. Bottom: Illustration of the Callan-Gross relation as a function of x for three Q^2 ranges.

3 The gluon and scaling violation

Suppose we say that, for a fixed value of Q^2, we define $u(x)$ and $\bar{u}(x)$ as the up and anti-up pdfs of the proton, respectively, and similarly define $d(x)$ and $\bar{d}(x)$ for the down quark[2]. Equation (12) then predicts the value of F_2 in terms of these pdfs:

$$F_2^p(x) = \frac{4}{9}x\{u(x) + \bar{u}(x)\} + \frac{1}{9}x\{d(x) + \bar{d}(x)\}. \tag{13}$$

The neutron is related to the proton by an isopin transformation, swapping u and d quarks. Assuming that the isospin symmetry is exact, we can therefore write F_2 for the neutron in terms of the proton pdfs:

$$F_2^n(x) = \frac{1}{9}x\{u(x) + \bar{u}(x)\} + \frac{4}{9}x\{d(x) + \bar{d}(x)\}. \tag{14}$$

Experimentally, the integrated F_2 values are found from low-energy (compared to the LHC) DIS with protons and deuterons:

$$\int_0^1 F_2^p(x)\,\mathrm{d}x = 0.18 \quad \text{and} \quad \int_0^1 F_2^n(x)\,\mathrm{d}x = 0.12. \tag{15}$$

From these, we can deduce that

$$\int_0^1 x\{u(x) + \bar{u}(x)\}\,\mathrm{d}x = 0.36 \quad \text{and} \quad \int_0^1 x\{d(x) + \bar{d}(x)\}\,\mathrm{d}x = 0.18. \tag{16}$$

Exercise: Verify this result, under the given assumptions. This is not as simple as it appears to be!

We now note that, as $u(x)$ is a probability density function, the product $xu(x)$ is a momentum density function, and similarly for the other pdfs. Therefore, we conclude from Equation (16) that up (anti)quarks carry approximately 36% of the proton's momentum, and down (anti)quarks carry about 18%. This leaves the remaining 46% of the proton's momentum unexplained. This is attributed to the presence of gluons, which cannot be directly measured in electroweak scattering experiments.

The gluons are, however, visible indirectly, through a process called gluon splitting. As a gluon propagates, it can spontaneously create a virtual quark-antiquark pair. Normally, they quickly annihilate, but the effect gives rise to non-zero *sea quark* pdfs, including those for strange, charm and even bottom quarks, and all varieties of antiquarks. It also affects the up and down quark pdfs. The gluon splitting rate is, however, highly dependent on both x and Q^2, and this breaks Björken scaling. As a result, all DIS quantities, including the pdfs, are generally quoted as a function of both x and Q^2.

[2] The antiquark densities are included for generality, but could be neglected for this discussion. We should in principle also account for other quark flavours, but for our purposes here this is not necessary.

4 Parton density functions

In the previous section, we saw how inclusive measurements of the structure functions can give hints as to the structure of hadrons. Combinations of more detailed differential analyses allow the individual pdfs to be calculated. We will focus on the most highly-studied hadron, the proton. Figure 4 shows measured parton density functions for three values of Q^2, spanning six orders of magnitude. In a non-interacting quark model, we would expect the pdfs to be of the form $\delta(x - \frac{1}{N})$ if there are N partons in the proton. Interactions between the quarks will smear this out and shift the peak to lower values of x, leading to a broad peak centred at $x \lesssim \frac{1}{N}$. We see that for values of Q^2 at or below m_p^2, the up and down quark pdfs do indeed peak at values of $x \sim 0.2 - 0.3$, and furthermore the pdfs are more or less (but not entirely) independent of Q^2. These peaks correspond to the *valence quarks* of the proton. There is still some evidence of scaling violation at these low values of Q^2, and a non-negligible antiquark component can also be seen.

In the lowest panel of Figure 4, a much higher value of Q^2 is shown, comparable to m_Z^2. Now the proton's momentum is dominated by gluons and sea quarks at low x. To balance this out, the valence pdfs at high x decrease slightly. The dominance of sea quarks in this final panel has important implications for modern hadron collider design. On-shell W and Z boson production occurs via quark-antiquark annihilation. The Tevatron had a beam energy of about 1 TeV, so that these bosons would be produced by partons with $x \sim \frac{80}{2000} = 0.04$. Production of more massive particles, such as a Higgs boson, would require higher values of x. In this region, the sea quark pdfs are still relatively small, and it is necessary to collide protons with antiprotons to obtain quark-antiquark collisions at sufficiently high rates for physics studies. The LHC, by contrast, was designed to collide 7 TeV proton beams. Now, a typical value of x for electroweak boson production is $\frac{80}{14000} \approx 0.005$. Here, the sea quark (including antiquark) pdfs are large, and so it is not necessary to collide antiprotons to produce these particles at appreciable rates. Therefore, the LHC can benefit from the relative ease of making high-density proton beams, achieving very high collision luminosities.

Exercise: In the above, a value of $x \sim m_W/(2E_b)$ has been used, where E_b is the beam energy. Show this to be a valid approximation, supposing that the partons from each incoming hadron have equal momentum fractions $x_1 = x_2 = x$, and they produce a W boson in the s-channel.

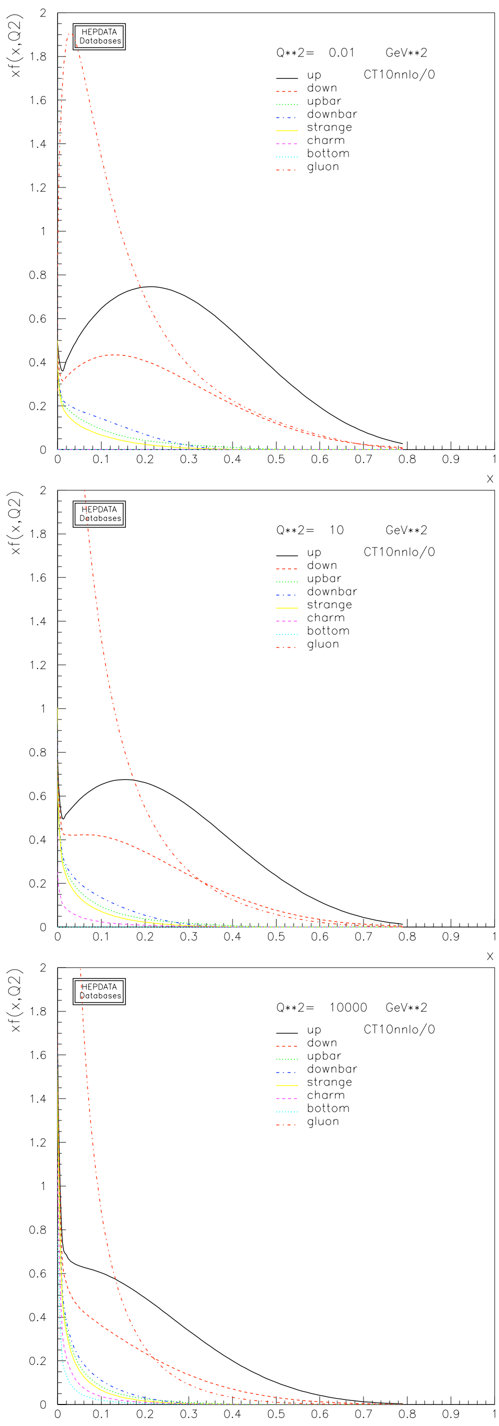

Figure 4: Parton density functions from a recent next-to-next-to-leading order fit to data from multiple experiments at three different Q^2 values. Top: $Q^2 = 0.01$ GeV^2, well below m_p^2. Middle: $Q^2 = 10$ GeV^2, larger than m_p^2. Bottom: $Q^2 = 10^4$ GeV^2, comparable to m_Z^2.