Addendum to tutorial 3: $\mathrm{U}(1)_Y \times \mathrm{SU}(2)_\mathrm{L}$ symmetry breaking

Dr. M Flowerdew

November 5, 2014

From an exercise in the first tutorial, we know that a generic $\mathrm{SU}(2)_\mathrm{L}$ transformation may be written in the following form:

$$G_{\mathrm{SU}(2)_\mathrm{L}} = \begin{pmatrix} \alpha & -\beta \\ \beta^* & \alpha^* \end{pmatrix}, \tag{1}$$

where α and β are arbitrary complex functions of space-time that satisfy $\alpha^*\alpha + \beta^*\beta = 1$. An arbitrary $\mathrm{U}(1)_Y$ transformation may be written as

$$G_{\mathrm{U}(1)_Y} = e^{iY\chi}, \tag{2}$$

where χ is an arbitrary function of space-time and Y is the hypercharge associated with a particular field. For the Higgs field, ϕ, defined to have a hypercharge of $Y = 1$, this reduces to $e^{i\chi}$, which we write as ϵ for convenience. Note that $\epsilon^*\epsilon = 1$.

Using Equations 1 and 2, we can write a complete (and, so far, arbitrary) $\mathrm{U}(1)_Y \times \mathrm{SU}(2)_\mathrm{L}$ transformation G, acting on the Higgs field:

$$\begin{aligned} G &= G_{\mathrm{U}(1)_Y} G_{\mathrm{SU}(2)_\mathrm{L}} \\ &= \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \begin{pmatrix} \alpha & -\beta \\ \beta^* & \alpha^* \end{pmatrix} \\ &= \begin{pmatrix} \epsilon\alpha & -\epsilon\beta \\ \epsilon\beta^* & \epsilon\alpha^* \end{pmatrix}. \end{aligned} \tag{3}$$

This is, in fact, the form for a generic $\mathrm{U}(2)$ transformation.

When breaking the electroweak symmetry, we wish to align the Higgs field vacuum state along a particular direction in $\mathrm{SU}(2)_\mathrm{L}$ space. We begin with the Higgs field in an arbitrary state

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} v_a \\ v_b \end{pmatrix}, \tag{4}$$

where v_a and v_b are complex functions of space-time. We wish to find the $\mathrm{U}(1)_Y \times \mathrm{SU}(2)_\mathrm{L}$ transformation which will transform ϕ to

$$\phi' = G\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \tag{5}$$

where $v^2 = v_a^2 + v_b^2$. Using the form of G in Equation (3), we can easily find the transformed field:

$$\begin{aligned} G\phi &= \frac{1}{\sqrt{2}} \begin{pmatrix} \epsilon\alpha & -\epsilon\beta \\ \epsilon\beta^* & \epsilon\alpha^* \end{pmatrix} \begin{pmatrix} v_a \\ v_b \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} \epsilon\alpha v_a - \epsilon\beta v_b \\ \epsilon\beta^* v_a + \epsilon\alpha^* v_b \end{pmatrix}. \end{aligned} \tag{6}$$

Requiring this to be equal to Equation (5), we can solve for α and β:

$$\alpha = \frac{v_b}{v}\epsilon; \qquad \beta = \frac{v_a}{v}\epsilon. \tag{7}$$

With these choices of α and β, the choice of gauge is fixed, and the electroweak symmetry is broken. Note that these values depend on ϵ – this means that the two original transformations are no longer independent, in other words it truly is the *combination* of both groups that is broken, rather than one or other individually.

It is instructive to write the resultant transformation G, using the values in Equation (7):

$$\begin{aligned} G &= \begin{pmatrix} \epsilon^2 \frac{v_b}{v} & -\epsilon^2 \frac{v_a}{v} \\ \frac{v_a^*}{v} & \frac{v_b^*}{v} \end{pmatrix} \\ &= \begin{pmatrix} \epsilon^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{v_b}{v} & -\frac{v_a}{v} \\ \frac{v_a^*}{v} & \frac{v_b^*}{v} \end{pmatrix}. \end{aligned} \tag{8}$$

The right-hand matrix in Equation (8) depends only on the initial Higgs field (Equation (4)), and is thus fixed entirely by the choice of gauge (Equation (5)). The left-hand matrix depends on the one remaining free parameter, ϵ. This indicates that the symmetry is not completely broken, there is still a form of $\mathrm{U}(1)_Y$ gauge symmetry, however it is not the same as $\mathrm{U}(1)_Y$ because the matrix containing it is not proportional to the $\mathrm{SU}(2)_\mathrm{L}$ identity matrix. Indeed, the lower component of the Higgs field is a gauge singlet of this new "electromagnetic" interaction, while by convention the upper component is assigned an electric charge of $Q = +1$. With this definition, we can write the transformation for a field with an arbitrary charge Q as

$$G_{\mathrm{U}(1)_Q} = e^{2iQ\chi} = \epsilon^{2Q}. \tag{9}$$

It is important that the matrix containing ϵ *pre*-multiplies the other matrix, as this means that it is applied to the fields *after* electroweak symmetry breaking.

Acting on a different multiplet, this new gauge interaction will in general have a different form. For example, the e_{L} field has a hypercharge of $Y = -1$, thus applying Equation (2) we note that

$$G^{e_{\mathrm{L}}}_{\mathrm{U}(1)_Y} = e^{-i\chi} = \epsilon^*. \tag{10}$$

The Higgs field and e_{L} are both $\mathrm{SU}(2)_{\mathrm{L}}$ doublets, and so Equation (1) applies equally to both. This means that the combined (gauge-fixed) $\mathrm{U}(1)_Y \times \mathrm{SU}(2)_{\mathrm{L}}$ transformation for e_{L} is

$$\begin{aligned} G &= \begin{pmatrix} \epsilon^* & 0 \\ 0 & \epsilon^* \end{pmatrix} \begin{pmatrix} \frac{v_b}{v}\epsilon & -\frac{v_a}{v}\epsilon \\ \frac{v_a^*}{v}\epsilon^* & \frac{v_b^*}{v}\epsilon^* \end{pmatrix} \\ &= \begin{pmatrix} \frac{v_b}{v} & -\frac{v_a}{v} \\ \frac{v_a^*}{v}\epsilon^{*2} & \frac{v_b^*}{v}\epsilon^{*2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \epsilon^{*2} \end{pmatrix} \begin{pmatrix} \frac{v_b}{v} & -\frac{v_a}{v} \\ \frac{v_a^*}{v} & \frac{v_b^*}{v} \end{pmatrix}. \end{aligned} \tag{11}$$

The right-hand matrix of the final line is identical to the right-hand matrix of Equation (8) – in other words, the same symmetry-breaking transformation is applied to both fields. The left-hand matrix is, however, different. In this case, it is the upper element of the e_{L} doublet (ν_e) that is a singlet of this new $\mathrm{U}(1)_Y$ group, while the lower element is electrically charged. Using Equation (9), we see that this particle has charge $Q = -1$.

Finally, we note that the e_{R} field transforms as an $\mathrm{SU}(2)_{\mathrm{L}}$ singlet and has a hypercharge of $Y = -2$. Therefore, under the symmetry-breaking transformation, this field transforms simply as $e^{i(-2)\chi} = \epsilon^{-2}$, and therefore also has an electric charge of $Q = -1$.

Exercise: Consider the symmetry-breaking transformation applied to an $\mathrm{SU}(2)_{\mathrm{L}}$ doublet of arbitrary hypercharge Y. Writing the $\mathrm{U}(1)_Y$ transformation as $G_{\mathrm{U}(1)_Y} = \epsilon^Y$, find the equivalent $\mathrm{U}(1)_Q$ transformation, equivalent to Equation (11). Wherever ϵ appears, compare with Equation (9) to find the electric charge of each state. Verify that your results are consistent with the expression $Q = Y/2 + I_3$.