Tutorial 4: Calculating cross sections and Feynman rules

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1 Preamble

Now that we have all the interactions of the Standard Model defined, we want to know how to use them to make predictions of the rates and properties of interactions between the different fields. In this tutorial, we will calculate the cross-section for one particular process using relativistic quantum mechanics, as an illustration of how this can be done. Along the way, we will verify that applying the Feyman rules for the corresponding diagram would yield the same result.

A general particle physics experiment is illustrated in Figure 1. This shows the initial state ψ_{in} being transformed into the final state ψ_{out} by some interaction within an experimental apparatus. For simplicity, we will assume that ψ_{in} and ψ_{out} can be modeled using plane wave momentum eigenstates, which only interact within a limited space-time volume VT. Outside that volume, all interactions are turned off.

Exercise: Why is it (mathematically) necessary to restrict the interaction volume when using plane waves?

Exercise: What physical justification can be made for the assumptions introduced in the above paragraph?

These states ψ_{in} and ψ_{out} are defined on boundary regions at times t_0 and t, respectively – eventually, we can imagine taking these to negative and positive infinity. The states propagate freely according to some Hamiltonian H_0, so that formally each wavefunction at any time t can be expressed as[1]

$$\psi(t) = e^{-iH_0(t-t_0)}\psi(t_0). \tag{1}$$

[1]For simplicity, we will assume that both ψ_{in} and ψ_{out} evolve according to H_0. In general they may not, for example when the collision is inelastic. This requires a more careful treatment of which basis states we use, but the essential points of the following arguments remain similar.

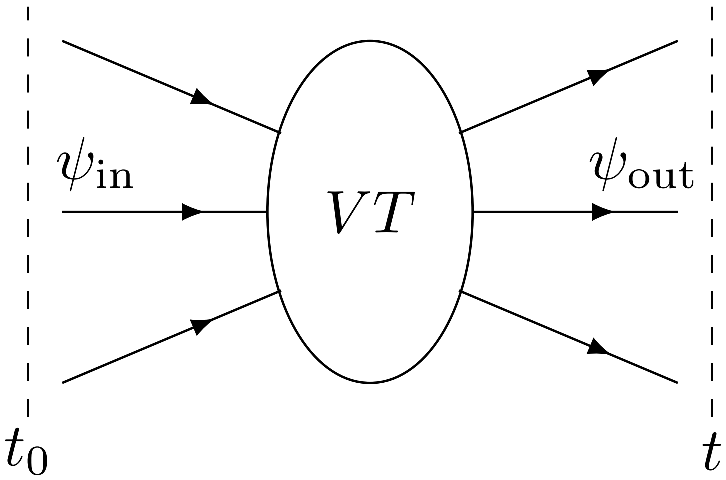

Figure 1: Schematic diagram of “in” and “out” states considered in high energy particle collisions.

This conventional approach to the evolution of wavefunctions is also called the *Schrödinger Picture*. For our purposes, it is however useful to use the *Interaction Picture* (or Dirac Picture) instead. In the Interaction Picture, the plane-wave time evolution is factored into the quantum mechanical operators, which allows changes due to interactions to be seen more clearly.

The Interaction Picture state is defined as follows:

$$\hat{\psi}(t) = e^{iH_0(t-t_0)}\psi(t). \tag{2}$$

Outside the interaction volume, these states therefore remain constant in time, and only evolve within the interaction volume VT.

Inside the volume, the Hamiltonian now has an extra term, H_1, describing the interaction(s). For example, an interaction between a fermion and the electromagnetic field is described by $H_1 = q\gamma^0\gamma^\mu A_\mu(x)$. The Schrödinger Picture wavefunction evolves according to

$$i\frac{\partial\psi}{\partial t} = H_0\psi + H_1\psi. \tag{3}$$

This can be used to derive the equation of motion for the Interaction Picture wavefunction $\hat{\psi}$:

$$\begin{aligned} i\frac{\partial\hat{\psi}}{\partial t} &= i\frac{\partial}{\partial t}\left(e^{iH_0(t-t_0)}\psi(t)\right) \\ &= i\left(\frac{\partial}{\partial t}e^{iH_0(t-t_0)}\right)\psi + ie^{iH_0(t-t_0)}\frac{\partial\psi}{\partial t} \\ &= -H_0e^{iH_0(t-t_0)}\psi + e^{iH_0(t-t_0)}\left(H_0\psi + H_1\psi\right) \\ &= e^{iH_0(t-t_0)}H_1\psi \\ &= \hat{H}_1\hat{\psi}, \end{aligned} \tag{4}$$

where

$$\hat{H}_1 = e^{iH_0(t-t_0)}H_1e^{-iH_0(t-t_0)} \tag{5}$$

and the terms involving H_0 cancel as H_0 trivially commutes with $e^{iH_0(t-t_0)}$.

1.1 Perturbation theory using the Interaction Picture

Now we have Equation (4), we can formally solve for $\hat{\psi}(t)$ within the interaction volume, assuming that H_1 is "small", such that we may use perturbation theory.

$$\hat{\psi}(t) = \hat{\psi}_{\text{in}}(t_0) - i\int_{t_0}^{t}\hat{H}_1(t_1)\hat{\psi}_{\text{in}}(t_1)\,\mathrm{d}t_1. \tag{6}$$

We can use this solution to replace $\hat{\psi}_{\text{in}}$ within the integral, and we see the beginnings of a perturbative expansion in $\hat{H}_1$:

$$\begin{aligned} \hat{\psi}(t) &= \hat{\psi}_{\text{in}}(t_0) - i\int_{t_0}^{t}\hat{H}_1(t_1)\left\{\hat{\psi}_{\text{in}}(t_0) - i\int_{t_0}^{t_1}\hat{H}_1(t_2)\hat{\psi}_{\text{in}}(t_2)\,\mathrm{d}t_2\right\}\mathrm{d}t_1 \\ &= \hat{\psi}_{\text{in}}(t_0) - i\int_{t_0}^{t}\hat{H}_1(t_1)\hat{\psi}_{\text{in}}(t_0)\,\mathrm{d}t_1 \\ &\quad + (-i)^2\int_{t_0}^{t}\int_{t_0}^{t_1}\hat{H}_1(t_1)\hat{H}_1(t_2)\hat{\psi}_{\text{in}}(t_0)\,\mathrm{d}t_2\,\mathrm{d}t_1 + \mathcal{O}(\hat{H}_1^3). \end{aligned} \tag{7}$$

Inverting Equations (2) and (5), we can obtain the expansion for the Schrödinger Picture wavefunction ψ_{in}:

$$\psi(t) = e^{-iH_0(t-t_0)}\psi_{\text{in}}(t_0) \tag{8a}$$

$$+ \int_{t_0}^{t} e^{-iH_0(t-t_1)}\left(-iH_1(t_1)\right)e^{-iH_0(t_1-t_0)}\psi_{\text{in}}(t_0)\,\mathrm{d}t_1 \tag{8b}$$

$$\begin{aligned} &+ \int_{t_0}^{t}\int_{t_0}^{t_1} e^{-iH_0(t-t_1)}\left(-iH_1(t_1)\right)e^{-iH_0(t_1-t_2)} \\ &\qquad \times\left(-iH_1(t_2)\right)e^{-iH_0(t_2-t_0)}\psi_{\text{in}}(t_0)\,\mathrm{d}t_2\,\mathrm{d}t_1 \end{aligned} \tag{8c}$$

$$+ \mathcal{O}(H_1^3).$$

These terms are understood in the following way, noting that H_1 and H_0 do not necessarily commute:

(8a) Evolution of the unperturbed state to time t, i.e. no interaction.

(8b) Evolution to time t_1, at which point an interaction occurs, after which the resulting state evolves freely to time t. This describes a single Feynman vertex.

(8c) Two interaction vertices, with free propagation between times t_2 and t_1.

Using the result of Equation (8), we can calculate the amplitude for a transition to any particular final state $\psi_{\text{out}}(t)$ by calculating the overlap integral $\langle \psi_{\text{out}}(t) | \psi(t) \rangle$ in the usual way. This we will proceed to do for one simple example.

2 Fermion-fermion scattering cross section

In this section, we compute the interaction cross section for the process shown in Figure 2. The two scattering fermions are assumed to be of different flavours, for example an electron and a muon, to avoid complications of multiple diagrams contributing to the same final state.

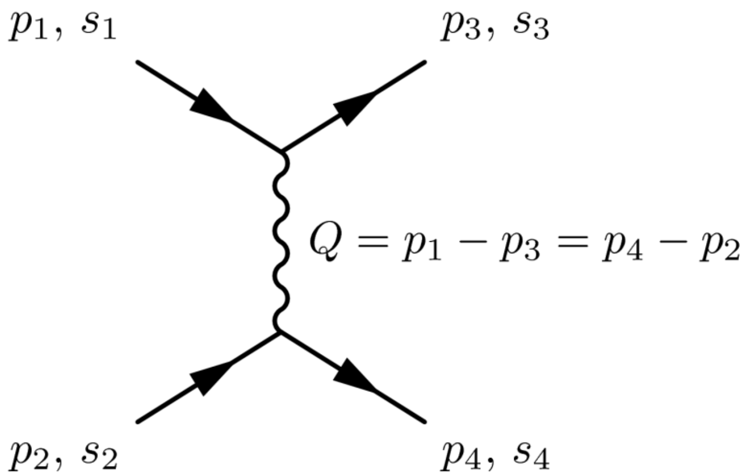

Figure 2: Diagram of first-order electromagnetic scattering of two non-identical fermions.

Each of the four fermion legs are characterised by their four-momentum p_i and spin s_i. Neglecting normalisation factors for now, the initial and final state wavefunctions (in the Schrödinger Picture) are

$$\psi_1(t_0) = u_{s_1}(p_1)e^{-i(E_1 t_0 - \boldsymbol{p_1}\cdot\boldsymbol{x})}, \qquad \psi_2(t_0) = u_{s_2}(p_2)e^{-i(E_2 t_0 - \boldsymbol{p_2}\cdot\boldsymbol{x})},$$
$$\psi_3(t) = u_{s_3}(p_3)e^{-i(E_3 t - \boldsymbol{p_3}\cdot\boldsymbol{x})}, \qquad \psi_4(t) = u_{s_4}(p_4)e^{-i(E_4 t - \boldsymbol{p_4}\cdot\boldsymbol{x})}. \tag{9}$$

Note that the $u_s(p)$ states are already in the correct form for the interaction Picture, i.e. with free-particle propagation removed. In much of the following, these will be written as u_1, u_2 etc., for simplicity.

Following the rules outlined in the previous section (in particular, Equation (8)), we can write down an expression for the amplitude of this transition, from the perspective of particle 1:

$$A = \int_{t_0}^{t} \langle\psi_3(t)|e^{-iH_0(t-t_1)}\left(-iH_1(t_1)\right)e^{-iH_0(t_1-t_0)}|\psi_1(t_0)\rangle\,\mathrm{d}t_1 \tag{10}$$

The exponentials involving H_0 act on the final and initial states to give $e^{-iE_3(t-t_1)}$ and $e^{-iE_1(t_1-t_0)}$, respectively. We will also substitute the appropriate form for H_1 for an electromagnetic interaction to give

$$\begin{aligned} A &= \int_{t_0}^{t}\int_V u_3^\dagger e^{i(E_3 t - \boldsymbol{p_3}\cdot\boldsymbol{x})} e^{-iE_3(t-t_1)}(-i)q_1\gamma^0\gamma^\mu A_\mu(\boldsymbol{x}, t_1) \\ &\qquad \times e^{-iE_1(t_1-t_0)} e^{-i(E_1 t_0 - \boldsymbol{p_1}\cdot\boldsymbol{x})} u_1\,\mathrm{d}^3\boldsymbol{x}\,\mathrm{d}t_1 \\ &= -iq_1\int_{t_0}^{t}\int_V u_3^\dagger\gamma^0\gamma^\mu u_1 A_\mu(\boldsymbol{x}, t_1)e^{-i[(E_1-E_3)t_1-(\boldsymbol{p_1}-\boldsymbol{p_3})\cdot\boldsymbol{x}]}\,\mathrm{d}^3\boldsymbol{x}\,\mathrm{d}t_1 \\ &= -iq_1\bar{u}_3\gamma^\mu u_1\int_{VT} A_\mu(x_1)e^{-i(p_1-p_3)x_1}\,\mathrm{d}^4x_1. \end{aligned} \tag{11}$$

The first part of this expression, $q_1\bar{u}_3\gamma^\mu u_1$, is interpreted as the fermion current for particle 1, such that the amplitude has the form $j_1^\mu A_\mu$. In the Interaction Picture, these state vectors no longer depend explicitly on space-time coordinates and can be removed from the integral. In terms of Feynman rules, also note that there are no "propagators" for external lines, as the phase factors associated with these have already been absorbed by the initial and final state vectors.

The remaining integrand becomes (in the limit where VT encompasses all of space-time) the Fourier transform of A_μ, where the argument $(p_1 - p_3)$ in the exponential ensures four-momentum conservation at the fermion-photon vertex.

2.1 The photon propagator and Green's functions

In this section, we will evaluate the photon propagator part of Equation (11).

$$\tilde{A}^\mu(p_1 - p_3) = \int_{VT} A^\mu(x_1)e^{-i(p_1-p_3)x_1}\,\mathrm{d}^4x_1. \tag{12}$$

This can be found by using the appropriate Green's function. The Green's function $G(x,x')$ associated with a differential operator $\mathcal{O}$ is defined as the function which satisfies the following equation

$$\mathcal{O}G(x,x') = \delta(x-x'). \tag{13}$$

This Green's function can be used to solve for any field satisfying the equation $\mathcal{O}\psi(x) = f(x)$, with the result $\psi(x) = \int G(x,x')f(x')\,\mathrm{d}x'$.

One interesting property of Equation (13) is that the Fourier transform of the delta function is 1, up to a constant factor that is equal on both sides of the equation. This means that the Fourier transform of the Green's function is just the inverse of $\mathcal{O}$, when $\mathcal{O}$ is expressed in momentum space variables. For example, for a fermion where $\mathcal{O} = i\gamma^\mu\partial_\mu - m = \not{p} - m$:

$$\begin{aligned}
(\not{p} - m)\tilde{G}(p) &= \int (i\gamma^\mu\partial_\mu - m)G(x,x')e^{-ip(x-x')}\,\mathrm{d}^4x' \\
&= \int \delta(x-x')e^{-ip(x-x')}\,\mathrm{d}^4x' \\
&= e^0 = 1 \\
\Rightarrow (\not{p}+m)(\not{p}-m)\tilde{G}(p) &= (\not{p}+m) \\
\text{or } \tilde{G}(p) &= \frac{(\not{p}+m)}{p^2-m^2}.
\end{aligned} \tag{14}$$

In our example, the photon field satisfies the equation $\Box A^\mu = j_2^\mu$, where j_2^μ is the current from particle 2. In this case, the operator $\mathcal{O}$ is the D'Alembertian $\Box = \partial^\mu\partial_\mu$, or Q^2 in momentum space if Q is the photon four-momentum. Therefore, the integral in Equation (12) may be written

$$\tilde{A}^\mu(Q) = \frac{1}{Q^2}\tilde{j}_2^\mu. \tag{15}$$

The Interaction Picture current j_2^μ is, by analogy with Equation (11), $q_2\bar{u}_4\gamma^\mu u_2$. Inserting this into Equation (15), and allowing for the extra phase factors from Equation (9), we find

$$\begin{aligned}
\tilde{A}^\mu(Q) &= \frac{1}{Q^2}\int_{VT} q_2\left(\bar{u}_4 e^{ip_4x_1}\right)\gamma^\mu\left(u_2e^{-ip_2x_1}\right)e^{-i(p_1-p_3)x_1}\,\mathrm{d}^4x_1 \\
&= \frac{g^{\mu\nu}}{Q^2}\int_{VT} q_2\bar{u}_4\gamma_\nu u_2 e^{-i(p_1+p_2-p_3-p_4)x_1}\,\mathrm{d}^4x_1 \\
&= \frac{g^{\mu\nu}}{Q^2}q_2\bar{u}_4\gamma_\nu u_2(2\pi)^4\delta^4(p_1+p_2-p_3-p_4).
\end{aligned} \tag{16}$$

Thus, we have another fermion current, the photon propagator $\frac{g^{\mu\nu}}{Q^2}$ and a delta function ensuring overall four-momentum conservation.

Putting this all together,we have the complete amplitude for this process:

$$
\begin{aligned}
A &= -(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\bar{u}_3(-iq_1\gamma^\mu)u_1\left(\frac{-ig_{\mu\nu}}{Q^2}\right)\bar{u}_4(-iq_2\gamma^\nu)u_2 \\
&= -(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)i\mathcal{M}_{fi}, \qquad (17)
\end{aligned}
$$

where $\mathcal{M}_{fi}$ is the *matrix element* for the $i \to f$ transition. Here, factors of i have been inserted where they are associated with vertex or propagator terms. These have no effect in this example, but are important when interference effects between multiple diagrams are considered.

Exercise: Verify that Equation (17) can also be obtained by applying the standard Feynman rules to the process in Figure 2.

2.2 Squaring the amplitude

The cross-section will, in the end, depend upon the squark of the amplitude just derived. For now, we will simply focus on the matrix element part of this calculation. From Equation (17):

$$
\begin{aligned}
|M_{fi}|^2 &= |\bar{u}_3(-iq_1\gamma^\mu)u_1\left(\frac{-i}{Q^2}\right)\bar{u}_4(-iq_2\gamma_\mu)u_2|^2 \\
&= \frac{q_1^2q_2^2}{Q^2}(\bar{u}_3\gamma^\mu u_1)(\bar{u}_4\gamma_\mu u_2)(\bar{u}_2\gamma^\nu u_4)(\bar{u}_1\gamma_\nu u_3) \\
&= \frac{q_1^2q_2^2}{Q^2}L_1^{\mu\nu}L_{2\mu\nu} \qquad (18)
\end{aligned}
$$

where

$$
L_1^{\mu\nu} = \bar{u}_3\gamma^\mu u_1\bar{u}_1\gamma^\nu u_3 \qquad (19)
$$

and similarly for $L_{2\mu\nu}$.

In the end, we wish to average over initial spin states and sum over final spin states. There are four initial spin states, as each incoming fermion has two possible helicities. Therefore, the matrix element we want to calculate is actually $\frac{1}{4}\sum_{\text{spins}}|M_{fi}|^2$. This can be evaluated using standard gamma matrix algebra, in particular using the following result (reinstating spin and momentum labels):

$$
\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m. \qquad (20)
$$

Thus, using α, β, ρ and σ as spinor indices:

$$\begin{aligned}\sum_{\text{spins}} L_1^{\mu\nu} &= \sum_{s_1,s_3} \bar{u}^{\alpha}_{s_3}(p_3)\gamma^{\mu,\alpha\beta}u^{\beta}_{s_1}(p_1)\bar{u}^{\rho}_{s_1}(p_1)\gamma^{\nu,\rho\sigma}u^{\sigma}_{s_3}(p_3) \\ &= \gamma^{\mu,\alpha\beta}\left[\not{p}_1 + m_1\right]^{\beta\rho}\gamma^{\nu,\rho\sigma}\left[\not{p}_3 + m_1\right]^{\sigma\alpha} \\ &= \mathrm{Tr}\left(\gamma^{\mu}(\not{p}_1 + m_1)\gamma^{\nu}(\not{p}_3 + m_1)\right) \\ &= \mathrm{Tr}\left(\gamma^{\mu}\not{p}_1\gamma^{\nu}\not{p}_3\right) + m_1^2\,\mathrm{Tr}\left(\gamma^{\mu}\gamma^{\nu}\right) \\ &= 4\left[p_1^{\mu}p_3^{\nu} - g^{\mu\nu}p_1\cdot p_3 + p_1^{\nu}p_3^{\mu} + m_1^2 g^{\mu\nu}\right]. \end{aligned} \tag{21}$$

The fermion tensor $L_{2\mu\nu}$ naturally has the same form.

After a little algebra, the final matrix element evaluates to the following:

$$\begin{aligned}\frac{1}{4}\sum_{\text{spins}}|M_{fi}|^2 = 4\frac{q_1^2q_2^2}{Q^4}&\left[2(p_1\cdot p_2)(p_3\cdot p_4) + 2(p_1\cdot p_4)(p_2\cdot p_3)\right. \\ &\left. -2m_1^2(p_2\cdot p_4) - 2m_2^2(p_1\cdot p_3) + 4m_1^2m_2^2\right]\end{aligned} \tag{22}$$

Exercise: Complete the missing steps in the above derivation.

Finally, we introduce the remaining Mandelstam variables, so that this result may be written in a more compact form:

$$\begin{aligned} s &= (p_1+p_2)^2 = (p_3+p_4)^2 \approx 2p_1\cdot p_2 \approx 2p_3\cdot p_4 \\ t &= (p_1-p_3)^2 = (p_2-p_4)^2 \approx -2p_1\cdot p_3 \approx -2p_2\cdot p_4 \\ u &= (p_1-p_4)^2 = (p_2-p_3)^2 \approx -2p_1\cdot p_4 \approx -2p_2\cdot p_3 \end{aligned} \tag{23}$$

In terms of these, Equation (22) becomes

$$\frac{1}{4}\sum_{\text{spins}}|M_{fi}|^2 = 2\frac{q_1^2q_2^2}{t^2}\left(s^2 + u^2 - (m_1^2+m_2^2)t + 4m_1^2m_2^2\right) \tag{24}$$

2.3 Fermi's Golden Rule

Now we have squared the matrix element, we need to consider what other information is required to calculate an interaction cross section. The full rate is given by Fermi's Golden Rule

$$\mathrm{d}\sigma = \frac{\frac{1}{VT}|A|^2\rho(E_f)\,\mathrm{d}E_f}{F}, \tag{25}$$

where $\rho(E_f)$ is the density of final states, division by VT normalises the interaction volume, and F is the particle flux.

Consider first the normalised square of the amplitude A. This will result in the product of two delta functions, which is not formally defined. However, we can proceed by noting that the second delta function will evaluate to $\delta^4(0)$ and will contribute a factor VT to the final result when integrated over space-time. In other words

$$\begin{aligned}\frac{1}{VT}|A|^2 &= \frac{1}{VT}(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\int_{VT} e^{-i\cdot 0}\,\mathrm{d}^4x|\mathcal{M}_{fi}|^2 \\ &= (2\pi)^4\delta^4(p_1+p_2-p_3-p_4)|\mathcal{M}_{fi}|^2. \end{aligned} \tag{26}$$

The Lorentz invariant phase space factor $\rho(E_f)\,\mathrm{d}E_f$ is defined as

$$\rho(E_f)\,\mathrm{d}E_f = \frac{1}{2E_3}\frac{1}{2E_4}\frac{\mathrm{d}^3\boldsymbol{p_3}}{(2\pi)^3}\frac{\mathrm{d}^3\boldsymbol{p_4}}{(2\pi)^3}. \tag{27}$$

We will evaluate this in the centre of momentum frame. Together with the delta function in Equation (26), the phase space integral over $\boldsymbol{p_4}$ evaluates to unity, leaving

$$\begin{aligned}\delta^4(p_1+p_2-p_3-p_4)\rho(E_f)\,\mathrm{d}E_f &\to \frac{1}{4(2\pi)^6}\delta(E_1+E_2-E_3-E_4)\frac{p_3^2\,\mathrm{d}p_3\,\mathrm{d}\Omega}{E_3E_4} \\ &= \frac{1}{4(2\pi)^6}\delta(E_1+E_2-E_3-E_4)\frac{p_3\,\mathrm{d}(E_3+E_4)}{E_3+E_4}\,\mathrm{d}\Omega \\ &\to \frac{1}{4(2\pi)^6}\frac{p_3}{E_1+E_2}\,\mathrm{d}\Omega \\ &\approx \frac{1}{8(2\pi)^6}\,\mathrm{d}\Omega. \end{aligned} \tag{28}$$

Here, we have used the fact that $\frac{\mathrm{d}(E_3+E_4)}{E_3+E_4} = \frac{p_3\mathrm{d}p_3}{E_3E_4}$, and the assumption that all particles are relativistic, so that $p_3 \approx E_3 = E_1 \approx E_2$.

Finally, there is the particle flux. This is the product of the particle densities (normalised to $2E$ particles per unit volume) and the difference of their velocities. In the relativistic limit, this becomes $4p_1 \cdot p_2 = 2s$, where s is the square of the energy in the centre of momentum frame.

Putting this together, we obtain the differential cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s}|\mathcal{M}_{fi}|^2. \tag{29}$$

Substituting in the result from Equation (24), and neglecting the particle masses, we obtain our final result

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{q_1^2q_2^2}{32\pi^2 s}\frac{s^2+u^2}{t^2}. \tag{30}$$