Tutorial 1: A quick review of relativistic quantum mechanics

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In this course, we will explore the Standard Model (SM) of particle physics using the language of relativistic quantum mechanics. The purpose of this first tutorial is to recap some of the nomenclature of this framework, and to explore the connection with Lagrangian formalisms. This last point is especially important, as the Lagrangian density (see below) is the point of connection between quantum mechanics and quantum field theory. The SM is more correctly expressed using quantum field theory, however this is very complicated to use. As we will see, relativistic quantum mechanics is sufficient for many purposes, so long as one takes care to respect its domain of validity. Quantum mechanics cannot properly describe the creation or annihilation of particles, for example. Through use of the Feynman rules, it is nevertheless possible to compute amplitudes for SM processes using the Lagrangian density, without any direct use of quantum field theory.

With this important caveat, we may proceed. Due to the connections with field theory, the terms "wave function" and "field" will often be used interchangeably. It is also assumed throughout that the reader is familiar with the Einstein summation convention and symbols such as x^μ and ∂_μ. Please refer to the lecture notes for a complete summary of the conventions used in this course.

1 Lagrangian mechanics

The classical Lagrangian L is a real function, defined in terms of the kinetic (T) and potential (V) energies of a system:[1]

$$L(\boldsymbol{x}, \dot{\boldsymbol{x}}) = T - V. \tag{1}$$

The equations of motion can be derived, if the Lagrangian is known, by considering variations in the action $S = \int L\,\mathrm{d}t$ and requiring that $\delta S = 0$.

[1]For the purpose of clarity, we will assume that the system has just one particle – conceptually the extension to multi-particle systems is straightforward.

This is the *prinicple of least action*. The variation δS can be found using the chain rule:

$$\begin{aligned}\delta S &= \int \delta L(\boldsymbol{x}, \dot{\boldsymbol{x}})\,\mathrm{d}t \\ &= \int \frac{\partial L}{\partial \boldsymbol{x}}\delta \boldsymbol{x} + \frac{\partial L}{\partial \dot{\boldsymbol{x}}}\delta \dot{\boldsymbol{x}}\,\mathrm{d}t \\ &= \int \frac{\partial L}{\partial \boldsymbol{x}}\delta \boldsymbol{x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\boldsymbol{x}}}\right)\delta \boldsymbol{x} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\boldsymbol{x}}}\delta \boldsymbol{x}\right)\,\mathrm{d}t. \end{aligned} \tag{2}$$

The final term of the last line, obtained using integration by parts, is a total derivative, and vanishes if we assume that the variation $\delta \boldsymbol{x}$ tends to zero as $t \to \pm\infty$. This leaves us with the first two terms, which are both proportional to $\delta \boldsymbol{x}$. If we require that $\delta S = 0$ for any arbitrary displacement, then the coefficient of $\delta \boldsymbol{x}$ must be identically zero, which yields the Euler-Lagrange equation

$$\frac{\partial L}{\partial \boldsymbol{x}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\boldsymbol{x}}} = 0. \tag{3}$$

Equation (3) and its many-particle extensions can describe systems of classical objects well, where their positions can be described in terms of precise coordinates. In contrast, quantum mechanical objects are described by their wavefunctions, with non-trivial spatial distributions. Therefore, the Lagrangian needs to be replced by a quantity that can depend on space as well as time, namely the *Lagrangian density*, $\mathcal{L}$. The Lagrangian density for a single particle is a real functional that depends on the wavefunction ψ and its space-time derivative $\partial_\mu \psi$.[2] The integral over time must also be replaced by a four-dimensional integral to obtain the action:

$$S = \int \mathcal{L}(\psi, \partial_\mu \psi)\,\mathrm{d}^4 x. \tag{4}$$

The equations of motion can now be derived in a similar way to Equation (2). A key conceptual difference is that the principle of least action can now be understood, as the contribution of S to the amplitude for a process is proportional to e^{iS}. Wave-function configurations close to where S is stationary will add coherently, while others will interfere destructively. Upon considering wave-function variations $\delta\psi$ that vanish at the space-time boundaries, we obtain the quantum-mechanical equivalent of Equation (3):

$$\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \psi\right)}\right) = 0. \tag{5}$$

The “equation of motion” in this case is the equation that describes the evolution of the wave function ψ.

[2] Note that we begin with a manifestly Lorentz-covariant notation, for later convenience.

Exercise 1: Derive Equation (5).

Answer: The derivation mirrors that of Equation (2):

$$\begin{aligned}\delta S &= \int \delta\mathcal{L}(\psi, \partial_\mu\psi)\,\mathrm{d}^4x \\ &= \int \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi\right)}\delta\left(\partial_\mu\psi\right)\,\mathrm{d}^4x \\ &= \int \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi\right)}\right)\delta\psi + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi\right)}\delta\psi\right)\,\mathrm{d}^4x.\end{aligned}$$

The last term is again a total derivative, and the remaining two terms are the left-hand side of Equation (5) multiplied by $\delta\psi$.

As with the classical Lagrangian, the Lagrangian density is a scalar quantity, and its symmetries can be related to conserved quantities via Noether's theorem. The symmetries of special relativity, for example, place stringent restrictions on the kind of terms that can be included in the Lagrangian density. We will return to the topic of symmetries in the SM throughout this course. For the moment, we will only consider the Lagrangian densities of *free*, i.e. non-inteacting, particles.

Formally, the Lagrangian can be recovered from the Lagrangian density via an integration over space:

$$L(t) = \int \mathcal{L}(\psi, \partial_\mu\psi)\,\mathrm{d}^3x. \tag{6}$$

However, as the value of the Lagrangian changes under Lorentz boosts it has limited use in particle physics.

Exercise 2: Consider the following Lagrangian density:

$$\mathcal{L} = i\psi^*\dot{\psi} - \frac{1}{2m}\nabla\psi^*\cdot\nabla\psi. \tag{7}$$

Derive the equation of motion for the wave function ψ by considering variations in ψ^* and $\nabla\psi^*$. Note that this Lagrangian density is not Lorentz invariant! You will need to account for this when applying the Euler-Lagrange equation.

Answer: This is not Lorentz invariant, so we need to separate the derivatives with respect to space and time.

$$\frac{\partial\mathcal{L}}{\partial\psi^*} = i\dot{\psi}, \qquad \frac{\partial\mathcal{L}}{\partial\dot{\psi}^*} = 0, \qquad \frac{\partial\mathcal{L}}{\partial(\nabla\psi^*)} = -\frac{1}{2m}\nabla\psi.$$

The non-relativistic version of Equation (5) is found to be

$$\frac{\partial \mathcal{L}}{\partial \psi^*} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}^*}\right) - \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial (\nabla \psi^*)}\right) = 0.$$

Therefore the equation of motion is easily found to be

$$i\dot{\psi} = -\frac{1}{2m}\nabla^2\psi,$$

in other words, the Schrödinger equation. We obtain the same answer if we consider variations in ψ:

$$\frac{\partial \mathcal{L}}{\partial \psi} = 0, \qquad \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\psi^*, \qquad \frac{\partial \mathcal{L}}{\partial (\nabla \psi)} = -\frac{1}{2m}\nabla\psi^*.$$

Substituting these into the equivalent Euler-Lagrange equation (the complex conjugate of the one above), we find

$$\begin{aligned} 0 - i\dot{\psi}^* + \frac{1}{2m}\nabla\psi^* &= 0 \\ \Rightarrow -i\dot{\psi}^* &= -\frac{1}{2m}\nabla\psi^*, \end{aligned}$$

which is the conjugate Schrödinger equation.

2 Scalar bosons

Consider a free real scalar field χ.[3] Its Lagrangian density may be written as[4]

$$\mathcal{L}_\chi = \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{1}{2}m^2\chi^2. \tag{8}$$

First we note the overall structure of the Lagrangian density. It consists of two terms:

- The *kinetic term* involving space-time derivatives of χ. In this case, the derivative is squared in order to create a Lorentz scalar. This is the quantum mechanical equivalent of T in Equation (1).
- The *mass term* proportional to the square of the field. This term has the opposite sign with respect to the kinetic term, in other words it can be regarded as a kind of potential, analogous to V in Equation (1).

[3] "Real" in this context means that the field is uncharged, e.g. the π^0. In this case, the particle χ and its antiparticle χ^* are identical entities.

[4] Equation (8) does not show *all* possible terms that can be constructed from χ and $\partial_\mu\chi$. In particular, a possible term proportional to χ^4 is neglected. This term would describe a self-interaction of the χ field, a subject we will return to later in the course.

As noted above, the forms of these terms are highly restricted by the requirements of special relativity. However, the overall normalisation is essentially arbitrary – the coefficient of $\frac{1}{2}$ in front of each term in Equation (8) is conventional. The relative sizes of the two terms *is* important, and this is parameterised by the parameter m.[5] These basic observations will be found to apply for all of the other particle types considered below.

The next step is to find the equation of motion for χ using the Euler-Lagrange equation:

$$\left(\partial^\mu \partial_\mu + m^2\right) \chi = 0. \tag{9}$$

This equation should already be familiar to you as the Klein-Gordan equation for a particle of mass m.

Exercise 3: Derive Equation (9) from Equations (5) and (8).

Answer: First we find the appropriate derivatives of $\mathcal{L}_\chi$.

$$\frac{\partial \mathcal{L}_\chi}{\partial \chi} = -m^2 \chi \qquad\qquad \frac{\partial \mathcal{L}_\chi}{\partial \left(\partial_\mu \chi\right)} = \partial^\mu \chi$$

Once this is done, we simply replace into Equation (5):

$$-m^2 \chi - \partial_\mu \partial^\mu \chi = 0.$$

Exercise 4: Show that

$$\chi = \frac{e^{-mr}}{r} \tag{10}$$

$$\text{and } \chi = A e^{-ip_\mu x^\mu} \tag{11}$$

are both valid solutions of Equation (9). What are the physical interpretations of these solutions?

Answer: For the first case, we need to consider the problem using sperical coordinates. The wave function χ does not depend on time or the angular coordinates θ and ϕ, so $\partial_\mu \partial^\mu \chi$ reduces to the radial

[5] Note that m must be real or pure imaginary, in order that m^2 is real.

component of $-\nabla^2\chi$:

$$\begin{aligned}
-\nabla^2\chi &= -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\chi}{\partial r}\right) \\
&= -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left[\frac{-mre^{-mr}-e^{-mr}}{r^2}\right]\right) \\
&= -\frac{1}{r^2}\frac{\partial}{\partial r}\left(-mre^{-mr}-e^{-mr}\right) \\
&= -\frac{1}{r^2}\left(-me^{-mr}+m^2re^{-mr}+me^{-mr}\right) \\
&= -\frac{1}{r^2}(m^2r)e^{-mr} \\
&= -m^2\chi
\end{aligned}$$

The expression on the last line trivially satisfies Equation (9). This solution to the Klein-Gordan equation could describe a classical potential well for an interaction mediated by the massive boson χ. In the case that $m = 0$ the potential varies as $1/r$, resulting in an inverse square law force.

In the second case, the wave function is a momentum eigenstate:

$$\begin{aligned}
\hat{p}^\mu\chi = i\partial^\mu\chi &= iA(-ip^\mu)e^{-ip_\mu x^\mu} \\
&= p^\mu\chi. \\
\Rightarrow \partial^\mu\partial_\mu\chi &= -p^\mu p_\mu\chi.
\end{aligned}$$

Assuming that the particle is on-shell, then $p^\mu p_\mu = m^2$ and it is evident that χ satisfies Equation (9). This is a solution for a freely propagating boson with momentum p^μ.

For a non-interacting *complex* scalar particle ϕ (e.g. $\pi^\pm$), the Lagrangian density is

$$\mathcal{L}_\phi = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi. \tag{12}$$

There are clear similarities between this and Equation (8). The relative factor of two between the coefficiencts is a result of the fact that a complex field has two real components, as seen in the following exercise. The naive approach of applying the Euler-Lagrange equation independently to the real and imaginary components of ϕ is valid, but algebraically tedious. One can obtain the equations of motion much more simply via the mathematical trick of treating ϕ and ϕ^* as the independent entities, in the same way as for Exercise 2.

Exercise 5: Derive Equation (12) from Equation (8) by considering the complex field ϕ to be composed of two real scalar fields ϕ_1 and ϕ_2:

$$\begin{aligned}\phi &= \frac{1}{\sqrt{2}}\left(\phi_1 + i\phi_2\right)\\ \phi^* &= \frac{1}{\sqrt{2}}\left(\phi_1 - i\phi_2\right)\end{aligned} \tag{13}$$

You may assume that $\mathcal{L}_\phi = \mathcal{L}_{\phi_1} + \mathcal{L}_{\phi_2}$.

Answer: The Lagrangian density for the two real scalar fields is

$$\mathcal{L}_\phi = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{1}{2}m^2\phi_1^2 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{2}m^2\phi_2^2.$$

Next, we consider some likely combinations of ϕ, ϕ^* and their derivatives:

$$\begin{aligned}\phi^*\phi &= \frac{1}{2}\left(\phi_1^2 + \phi_2^2\right)\\ \partial_\mu\phi &= \frac{1}{\sqrt{2}}\left(\partial_\mu\phi_1 + i\partial_\mu\phi_2\right)\\ \Rightarrow \partial_\mu\phi^*\partial^\mu\phi &= \frac{1}{2}\left(\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2\right)\end{aligned}$$

It is clear by inspection that we can use these relationships to replace terms in $\mathcal{L}_\phi$ to obtain Equation (12).

Exercise 6: Evaluate the derivatives of $\mathcal{L}_\phi$ with respect to ϕ^* and $\partial_\mu\phi^*$, assuming that ϕ and $\partial_\mu\phi$ are constant. Use these to determine the equation of motion for ϕ. What happens if you take derivatives of $\mathcal{L}_\phi$ with respect to ϕ instead of ϕ^*? Compare both results to Equation (9).

Answer:

$$\begin{aligned}\frac{\partial\mathcal{L}_\phi}{\partial\phi^*} &= -m^2\phi,\\ \text{and } \frac{\partial\mathcal{L}_\phi}{\partial\left(\partial_\mu\phi^*\right)} &= \partial^\mu\phi.\\ \text{Thus: } -m^2\phi - \partial_\mu\partial^\mu\phi &= 0.\end{aligned}$$

The field ϕ satisfies the Klein-Gordan equation.

Taking derivatives with respect to ϕ, we find that ϕ^* also satisfies the Klein-Gordan equation:

$$\frac{\partial \mathcal{L}_\phi}{\partial \phi} = -m^2 \phi^*,$$
$$\text{and } \frac{\partial \mathcal{L}_\phi}{\partial \left(\partial_\mu \phi\right)} = \partial^\mu \phi^*.$$
$$\Rightarrow -m^2 \phi^* - \partial_\mu \partial^\mu \phi^* = 0.$$

In either case, the Lagrangian density contains all information relevant to the system, just as in classical mechanics. In particular, conserved quantities can be inferred from it, often by inspection. For example, it is clear that Equation (12) is invariant under the transformation $\phi \to e^{i\theta}\phi$, and this fact is related to the global conservation of electric charge.

Exercise 7: Verify that $\partial_\mu j^\mu = 0$ if $j^\mu \propto (\partial^\mu \phi^*)\, \phi - (\partial^\mu \phi)\, \phi^*$. Comment on the fact that j^μ vanishes if ϕ is real.

Answer: We obtain the answer by simple substitution and use of Equation (9):

$$\begin{aligned}
\partial_\mu \left[(\partial^\mu \phi^*)\, \phi - (\partial^\mu \phi)\, \phi^* \right] &= (\partial_\mu \partial^\mu \phi^*)\, \phi + (\partial^\mu \phi^*)\, (\partial_\mu \phi) \\
&\quad - (\partial_\mu \partial^\mu \phi)\, \phi^* - (\partial^\mu \phi)\, (\partial_\mu \phi^*) \\
&= -m^2 \phi^* \phi + m^2 \phi \phi^* \\
&= 0.
\end{aligned}$$

The quantity j^μ is a conserved current (and can be related to electric current). If ϕ is real then the particle has no charge and therefore its current is not conserved.

3 Fermions

The Lagrangian density for a free fermion field ψ is

$$\mathcal{L}_\psi = \bar{\psi} \left(i\gamma^\mu \partial_\mu - m \right) \psi, \tag{14}$$

where γ^μ are the *gamma matrices* introduced in the lectures and discussed below, and $\bar{\psi} = \psi^\dagger \gamma^0$ is the adjoint spinor to ψ. The corresponding equation of motion is the Dirac equation

$$\left(i\gamma^\mu \partial_\mu - m \right) \psi = 0. \tag{15}$$

Exercise 8: Derive the Dirac equation from Equation (14).

Answer: This is trivial upon inspection of the derivatives:

$$\frac{\partial \mathcal{L}_\psi}{\partial \bar{\psi}} = \left(i\gamma^\mu \partial_\mu - m\right)\psi, \qquad \frac{\partial \mathcal{L}_\psi}{\partial\left(\partial_\mu \bar{\psi}\right)} = 0.$$

To make sure this isn't a trick, let's try a different approach. Integrating Equation (15) by parts gives us

$$\mathcal{L}_\psi = i\partial_\mu\left(\bar{\psi}\gamma^\mu\psi\right) - i\left(\partial_\mu\bar{\psi}\right)\gamma^\mu\psi - m\bar{\psi}\psi.$$

The first term is a total derivative and is irrelevant as long as $\bar{\psi}\gamma^\mu\psi \to 0$ at infinity. Next, we take the usual derivatives:

$$\frac{\partial \mathcal{L}_\psi}{\partial \bar{\psi}} = -m\psi, \qquad \frac{\partial \mathcal{L}_\psi}{\partial\left(\partial_\mu \bar{\psi}\right)} = -i\gamma^\mu\psi.$$

From this we can write the equation of motion:

$$-m\psi + i\partial_\mu\gamma^\mu\psi = 0.$$

As with the examples in Section 2, Equation (14) consists of a kinetic term and a mass term. Unlike for scalar particles, the involvement of the gamma matrices mean that is possible for the Lagrangian density to be linear in the space-time derivatives.

The properties of the gamma matrices are discussed at length in the lecture notes. However, it is interesting to see that their properties can be deduced directly from the Dirac equation by applying the operator $-i\gamma^\mu\partial_\mu - m$:

$$\begin{aligned}\left(-i\gamma^\mu\partial_\mu - m\right)\left(i\gamma^\nu\partial_\nu - m\right)\psi &= \left(-i\gamma^\mu\partial_\mu - m\right)0 \\ \Rightarrow \left(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2\right)\psi &= 0. \end{aligned} \tag{16}$$

In order to be consistent with the Klein-Gordan equation, the first term must equal $\partial^\mu\partial_\mu$. This would appear to require that $\gamma^\mu\gamma^\nu = g^{\mu\nu}$, which is impossible. However, the fact that $\partial_\mu\partial_\nu$ is manifestly symmetric in the indices μ and ν means that we can replace the product of gamma matrices by the symmetrised version $\frac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)$. By requiring that this is equal to $g^{\mu\nu}$, we then obtain the gamma matrix algebra:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{17}$$

In addition, the Hermitian conjugate matrices are usually constrained to satisfy

$$\gamma^{0\dagger} = \gamma^0 \text{ and } \gamma^{i\dagger} = -\gamma^i, \tag{18}$$

where $i \in \{1, 2, 3\}$. A further matrix, called γ^5, is also defined:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{19}$$

Exercise 9: Show that γ^5 is Hermitian, that $(\gamma^5)^2 = 1$, and that it anticommutes with the other gamma matrices.

Answer: First, to show that γ^5 is Hermitian:

$$\begin{aligned}\gamma^{5\dagger} = -i\gamma^{3\dagger}\gamma^{2\dagger}\gamma^{1\dagger}\gamma^{0\dagger} &= i\gamma^3\gamma^2\gamma^1\gamma^0 \\ &= -i\gamma^2\gamma^1\gamma^0\gamma^3 \\ &= -i\gamma^1\gamma^0\gamma^2\gamma^3 \\ &= i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5.\end{aligned}$$

Next, let's square it:

$$\begin{aligned}(\gamma^5)^2 &= -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 \\ &= (\gamma^0)^2\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 \\ &= (\gamma^0)^2(\gamma^1)^2\gamma^2\gamma^3\gamma^2\gamma^3 \\ &= -(\gamma^0)^2(\gamma^1)^2(\gamma^2)^2(\gamma^3)^2.\end{aligned}$$

Recalling Equation (17), it is clear that $(\gamma^0)^2 = 1$, while $(\gamma^i)^2 = -1$ for $i \in \{1, 2, 3\}$. Putting it all together, we have the result that $(\gamma^5)^2 = 1$.

Finally, we want to obtain a value for $\{\gamma^5, \gamma^\mu\}$. Consider the product $\gamma^5\gamma^0$. We can turn this into $\gamma^0\gamma^5$ by commuting the γ^0 matrix with each component matrix of γ^5 in turn. It anticommutes with three of the component matrices, while naturally it commutes with itself. Therfore the entire operation picks up a minus sign, i.e.:

$$\gamma^5\gamma^0 = -\gamma^0\gamma^5.$$

The same result is obtained for the other gamma matrices.

The above definitions still leave some freedom in precise form (or basis) of the gamma matrices. Two common choices are the Dirac basis:[6]

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{20}$$

and the Weyl basis:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{21}$$

As will be seen in the exercises below, the Dirac basis can be related to the fermion's spin eigenstates (i.e. *helicity*). As we will see below, the Weyl basis is related to *chirality*, which is the limit of helicity in the limit that $p \to \infty$.

[6] The symbol "1" is understood to refer to the 2×2 unit matrix where appropriate.

Exercise 10: The lecture notes give explicit solutions for ψ in the Dirac basis, labeled $u_{1,2}$ and $v_{1,2}$. Choose one solution and show that it is an eigenvector of helicity in the case where $p_z = p$ and $p_x = p_y = 0$ (Bonus question: why is this assignment necessary?). The helicity operator is

$$\begin{pmatrix} \boldsymbol{\sigma} \cdot \boldsymbol{p} & 0 \\ 0 & \boldsymbol{\sigma} \cdot \boldsymbol{p} \end{pmatrix} = \begin{pmatrix} p_z & p_x - ip_y & 0 & 0 \\ p_x + ip_y & -p_z & 0 & 0 \\ 0 & 0 & p_z & p_x - ip_y \\ 0 & 0 & p_x + ip_y & -p_z \end{pmatrix}. \tag{22}$$

Answer: The Pauli spin matrices explicitly assume that spin is quantised along the z axis. The helicity is the projection of spin along the direction of motion, and therefore this is only well-defined if the direction of motion is also along the z axis. The solutions to the Dirac equation are also rather simple. Take for example u_1:

$$u_1 = \sqrt{\frac{E+m}{V}} \begin{pmatrix} 1 \\ 0 \\ \frac{p}{E+m} \\ 0 \end{pmatrix}$$

In addition, Equation (22) simplifies down to

$$\begin{pmatrix} \sigma_z p & 0 \\ 0 & \sigma_z p \end{pmatrix} = p \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

It is straightforward to see that, in this case, u_1 is an eigenvector of the helicity operator with eigenvalue p. Similar results hold for u_2, v_1 and v_2.

The chirality projection operators are defined as follows:

$$P_{\mathrm{L}} = \frac{1}{2}\left(1 - \gamma^5\right), \qquad P_{\mathrm{R}} = \frac{1}{2}\left(1 + \gamma^5\right). \tag{23}$$

These satisfy the relationship $P_{\mathrm{L}} + P_{\mathrm{R}} = 1$, and so we can always write a fermionic spinor as a sum of left- and right-handed components

$$\psi = (P_{\mathrm{L}} + P_{\mathrm{R}})\psi = \psi_{\mathrm{L}} + \psi_{\mathrm{R}}. \tag{24}$$

Exercise 11: Use Equation (20) to find an explicit form for P_{L} and P_{R} in the Dirac basis. Use this together with the Dirac equation solutions to compute $P_{\mathrm{L}}u_1$ and $P_{\mathrm{R}}u_1$. Verify that these behave as expected in the limits of $p_z \to 0$ and $p_z \to \pm\infty$. As in the last exercise, assume that $p_x = p_y = 0$.

Answer: First, let's find the chirality projection operators in the Dirac basis.

$$P_{\mathrm{L}} = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \qquad P_{\mathrm{R}} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Then we find:

$$\begin{aligned} P_{\mathrm{L}}u_1 &= \sqrt{\frac{E+m}{4V}}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ \frac{p}{E+m} \\ 0 \end{pmatrix} \\ &= \sqrt{\frac{E+m}{4V}}\begin{pmatrix} 1-\frac{p}{E+m} \\ 0 \\ -1+\frac{p}{E+m} \\ 0 \end{pmatrix} \\ &= \frac{1}{\sqrt{4V(E+m)}}\begin{pmatrix} E+m-p \\ 0 \\ p-E-m \\ 0 \end{pmatrix}. \end{aligned}$$

This vanishes in the limit that $p \to \infty$ (equivalently, $m \to 0$). In the case $p = 0$ ($E = m$), it becomes

$$P_{\mathrm{L}}u_1 = \frac{1}{\sqrt{8Vm}}\begin{pmatrix} 2m \\ 0 \\ -2m \\ 0 \end{pmatrix} = \sqrt{\frac{m}{2V}}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}.$$

The right-handed projection is

$$
\begin{aligned}
P_{\mathrm{R}} u_1 &= \sqrt{\frac{E+m}{4V}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \frac{p}{E+m} \\ 0 \end{pmatrix} \\
&= \sqrt{\frac{E+m}{4V}} \begin{pmatrix} 1 + \frac{p}{E+m} \\ 0 \\ 1 + \frac{p}{E+m} \\ 0 \end{pmatrix} \\
&= \frac{1}{\sqrt{4V(E+m)}} \begin{pmatrix} E+m+p \\ 0 \\ E+m+p \\ 0 \end{pmatrix} .
\end{aligned}
$$

In the limit that $p \to \infty$, this becomes

$$
P_{\mathrm{R}} u_1 = \frac{1}{\sqrt{4VE}} \begin{pmatrix} 2E \\ 0 \\ 2E \\ 0 \end{pmatrix} = \sqrt{\frac{E}{V}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} ,
$$

while in the case of $p = 0$, we find

$$
P_{\mathrm{R}} u_1 = \frac{1}{\sqrt{8Vm}} \begin{pmatrix} 2m \\ 0 \\ 2m \\ 0 \end{pmatrix} = \sqrt{\frac{m}{2V}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} .
$$

These results are as we would expect. With infinite momentum, the u_1 state (which has positive helicity) is fully right-handed, while the helicity and chirality are uncorrelated when $p = 0$.

In the Weyl basis, the helicity projection operators take an especially simple form as γ^5 is diagonal:

$$
P_{\mathrm{L}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} , \qquad P_{\mathrm{R}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} . \tag{25}
$$

Therefore, in this basis, we can write the spinor ψ as

$$
\psi = \begin{pmatrix} \psi'_{\mathrm{L}} \\ \psi'_{\mathrm{R}} \end{pmatrix} , \tag{26}
$$

where ψ'_{L} and ψ'_{R} are two-component Weyl spinors.

4 Vector bosons

In the Standard Model, gauge bosons are described by spin-1 (or *vector*) fields. The canonical example of this is the electromagnetic field, described by the four-vector potential A^μ. If we generalise this example to the case of a massive vector boson, the corresponding Lagrangian density is

$$\mathcal{L}_A = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2 A^\mu A_\mu, \tag{27}$$

$$\text{where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{28}$$

In the case of electromagnetism, the elements of $F^{\mu\nu}$ correspond to components of the electric and magnetic field vectors. As in the previous examples, the mass term is quadratic in the field A^μ, and its sign is opposite to the kinetic term. The remaining complexity with respect to Equation (8) mostly arises from the requirement of Lorentz invariance when applied to a vector field rather than a scalar. As in the other cases, the Euler-Lagrange equation can be used to derive the equation of motion for A^μ.

Exercise 12: Use Equation (27) to derive the Proca equation

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0. \tag{29}$$

Note that this reduces to $\partial_\mu F^{\mu\nu} = 0$ in the case that $m = 0$, consistent with Maxwell's equations.

Answer: The derivative of $\mathcal{L}_A$ with respect to A^μ is trivial:

$$\frac{\partial \mathcal{L}_A}{\partial A_\mu} = m^2 A^\mu.$$

The derivative with respect to $\partial_\mu A^\nu$ is tricky to get right. First let's expand the kinetic term of Equation (27):

$$\begin{aligned}
-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} &= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\
&= -\frac{1}{4}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu - \partial_\nu A_\mu \partial^\mu A^\nu + \partial_\nu A_\mu \partial^\nu A^\mu) \\
&= -\frac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu),
\end{aligned}$$

once we remember that the μ and ν indices may be freely swapped within an individual term. The derivative of the first term is straightforward:

$$-\frac{1}{2}\frac{\partial}{\partial\left(\partial_\mu A_\nu\right)}(\partial_\mu A_\nu \partial^\mu A^\nu) = -\partial^\mu A^\nu.$$

The derivative of the second term is harder to see and requires some manipulation of the indices:

$$
\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial\left(\partial_\mu A_\nu\right)}(\partial_\mu A_\nu \partial^\nu A^\mu) &= \frac{1}{2}\left(\frac{\partial}{\partial\left(\partial_\mu A_\nu\right)}(\partial_\mu A_\nu)\partial^\nu A^\mu + \partial_\mu A_\nu \frac{\partial}{\partial\left(\partial_\mu A_\nu\right)}\left(\partial^\nu A^\mu\right)\right) \\
&= \frac{1}{2}\left(\partial^\nu A^\mu + \partial^\nu A^\mu \frac{\partial}{\partial\left(\partial_\mu A_\nu\right)}\partial_\mu A_\nu\right) \\
&= \partial^\nu A^\mu .
\end{aligned}
$$

Therefore, the complete term is

$$
\begin{aligned}
\frac{\partial}{\partial\left(\partial_\mu A_\nu\right)}\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right) &= -\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) \\
&= -F^{\mu\nu}.
\end{aligned}
$$

The Proca equation follows trivially from these results.

Exercise 13: Show that the Proca equation is satisfied by a wavefunction of the form

$$
A^\mu = \varepsilon^\mu e^{ik_\nu x^\nu}. \tag{30}
$$

What condition must ε^μ satisfy? Confirm that the polarisation vectors given in the lecture notes satisfy this constraint.

Answer: First let's find $F^{\mu\nu}$ for Equation (30).

$$
F^{\mu\nu} = i\left(k^\mu \varepsilon^\nu - k^\nu \varepsilon^\mu\right) e^{ik_\kappa x^\kappa}.
$$

Next, we substitute this in to Equation (29):

$$
\begin{aligned}
\partial_\mu F^{\mu\nu} + m^2 A^\nu &= i\left(k^\mu \varepsilon^\nu - k^\nu \varepsilon^\mu\right) \cdot ik_\mu e^{ik_\kappa x^\kappa} + m^2 \varepsilon^\nu e^{ik_\kappa x^\kappa} \\
&= \left(-k^\mu k_\mu \varepsilon^\nu + k^\nu k^\mu \varepsilon_\mu + m^2 \varepsilon^\nu\right) e^{ik_\kappa x^\kappa}.
\end{aligned}
$$

The first and third terms cancel, as $k^\mu k_\mu = m^2$. Thus, the Proca equation is satisfied if and only if $k^\mu \varepsilon_\mu = 0$.

We now confirm this for the specific polarisation vectors in the lecture notes.

$$
\varepsilon_\mu(k, \pm 1)k^\mu = \mp\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & \pm i & 0 \end{pmatrix}\begin{pmatrix} E \\ 0 \\ 0 \\ k \end{pmatrix} = 0.
$$

$$\varepsilon_\mu(k,0)k^\mu = \mp\frac{1}{m}\begin{pmatrix}k & 0 & 0 & E\end{pmatrix}\begin{pmatrix}E\\0\\0\\k\end{pmatrix} = \mp\frac{1}{m}\left(kE - Ek\right) = 0.$$