Tutorial 2: Groups and symmetry

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1 What is a group?

A group is a set of objects (*elements* or *members*), which can be combined by some operation (addition, multiplication, etc.). For a description of the Standard Model (SM), it is only necessary to consider groups under multiplication, and therefore the notation of multiplication will be used for simplicity from the start. When applying this operation, four conditions must be satisfied:

1. For all elements a, b in the group, the combination ab is also a member of the group.
2. The operation must be associative, i.e. $(ab)c = a(bc)$.
3. There is an identity element e, such that $ae = ea = a$ for all elements.
4. Every element a has an inverse a^{-1}, such that $aa^{-1} = a^{-1}a = e$.

The identity element is commonly written e for generality. For multiplicative groups, e is just the number 1, or an appropriate identity matrix. In contrast, the identity element for addition is 0. From now on, "1" will replace e as labeling the identity element.

1.1 Examples

The simplest group is the trivial group:

$$\{1\}. \tag{1}$$

This has only one member, and yet satisfies all of the properties required of a group (under multiplication). We can add one member to this to construct a simple non-trivial group:

$$\{1, -1\}. \tag{2}$$

Here, each element is its own inverse.

We can further extend this, to construct a four-element complex group:

$$\{1, -1, i, -i\}. \tag{3}$$

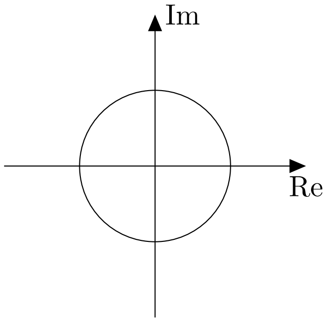
The new elements, i and $-i$, are inverses of each other. Note that this contains $\{1, -1\}$ as a *subgroup*.

Exercise 1: Show that this set of four matrices forms a group:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}. \tag{4}$$

Verify that the group structure is identical to the group of Equation (3).

As an example of a continuous group under multiplication, take the set of complex numbers with modulus 1. This group is denoted U(1), a name which derives from the fact that this is a *unitary* group with one dimension. In the complex plane, the members of this group trace out the unit circle.



This group plays an important role in physics, as it is a part of the gauge group of the Standard Model. It is combined with other groups via group multiplication. The product of two groups is a difficult concept to fully grasp, but can be understood by considering the product $\mathrm{U}(1) \times \mathbb{R}_{>0}$, where $\mathbb{R}_{>0}$ denotes the set of all positive real numbers. This yields the set of nonzero complex numbers $\mathbb{Z}_{\neq 0}$ with elements z:

$$z = re^{i\phi}, \qquad r \in \mathbb{R}_{>0}, \phi \in \mathbb{R}. \tag{5}$$

Thus, the group theoretic product of the unit circle and an semi-infinite radial axis spans the complex plane,[1] including elements that are in neither group individually. Note also that U(1) and $\mathbb{R}_{>0}$ are therefore subgroups of $\mathbb{Z}_{\neq 0}$.

All of these continuous groups are of a particular form, called *Lie groups*, which we will now examine.

[1]Except zero, which has no finite multiplicative inverse. With zero included, $\mathbb{Z}$ is a group under addition.

2 Lie groups

The defining property of a Lie group is that all elements can be reached by successive infinitesimal steps, usually starting from the identity element. Consider a member $(1 + i\epsilon)$ of U(1) a small distance ϵ from the identity. When thought of in terms of transformations, this corresponds to a small (eventually infinitesimal) rotation in the complex plane. Rewriting ϵ as α/N, where N is a large integer, we can imagine applying this small rotation N times. Mathematically, we achieve this by multiplying $(1 + i\alpha/N)$ by itself N times, i.e. by computing $(1 + i\alpha/N)^N$. Taking the limit as $N \to \infty$, we obtain a generic member of the group:

$$\lim_{N\to\infty} \left(1 + i\frac{\alpha}{N}\right)^N = e^{i\alpha}. \tag{6}$$

This is therefore a Lie group, as only an infinitesimally small region around the identity element needs to be known in order to characterise the entire group.

Due to this property, Lie groups are often characterised in terms of their *generators*, which in our case can be thought of as unit vectors describing possible directions in which transformations can be made. The number of these directions is called the *dimension* n of the group. With a collection of generators $\boldsymbol{T}$, and associated parameters $\boldsymbol{\alpha}$ (again, one for each dimension), a generic Lie group member is written in exponential notation like Equation (6):

$$\lim_{N\to\infty} \left(1 + i\frac{\boldsymbol{\alpha}\cdot\boldsymbol{T}}{N}\right)^N = e^{i\boldsymbol{\alpha}\cdot\boldsymbol{T}}. \tag{7}$$

This can be taken to be a definition of what we mean by $e^{i\boldsymbol{\alpha}\cdot\boldsymbol{T}}$.

> **Exercise 2:** Comparing Equations (6) and (7), what is the generator for U(1)?

When there is more than one generator, it is important to ask whether or not different members of the group will *commute*. The same observation applies to the generators of a Lie group. The commutators of the group generators define the *algebra* of the group:

$$[T_a, T_b] = T_aT_b - T_bT_a = if_{abc}T_c. \tag{8}$$

Note that the right-hand side is linear in the group generators; this is a consequence of all products of group members also being in the group.

The numbers f_{abc} are called the group's *structure constants*. If all are zero, then the group's generators (and elements) all commute; the group is *Abelian*. U(1) is an Abelian group, as all complex numbers with modulus one commute with each other. Multi-dimensional groups may be *non-Abelian*. When applied to field theories, these groups lead to self-interacting gauge fields.

3 A first look at SU(2)

The name SU(2) refers to the group of *special* 2×2 unitary matrices. The term "special" means that these matrices have determinant 1, thus preserving the normalisation of state vectors upon which they act. As with any Lie group, any member of SU(2), G, can be written in terms of the group's generators $\boldsymbol{T}$

$$G = e^{i\boldsymbol{\alpha}\cdot\boldsymbol{T}}. \tag{9}$$

The generators of SU(2) are familiar, as they are proportional to the Pauli spin matrices:

$$T_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad T_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad T_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{10}$$

These are Hermitian matrices with zero trace, which ensure that the group elements in Equation (9) are unitary with unit determinant. Correspondingly, $\boldsymbol{\alpha}$ is a three-component vector in the associated Hilbert space, and the group has dimension 3. In addition, the generator matrices have the following important properties:

$$[T_a, T_b] = i\epsilon_{abc}T_c, \text{ and } T_a^2 = \frac{1}{4}\boldsymbol{I}. \tag{11}$$

> **Exercise 3:** Express $\boldsymbol{\alpha}\cdot\boldsymbol{T}$ as a 2×2 matrix. Use the Taylor series expansion of Equation (9) to find G. Does it have the expected properties? What value of $|\boldsymbol{\alpha}|$ corresponds to a full rotation?

The version of G just derived is expressed in the so-called *fundamental representation* of SU(2). In other words, the generator matrices have the minimum possible size (2×2) that can possibly express the group algebra of Equation (11). Other, larger, matrices can also satisfy the group algebra, leading to alternative *representations* of the group. The *adjoint* representation is especially important for our purposes, when the dimension of the generator matrices matches the dimension of the group. Both of these representations will be explored in the next section.

3.1 Representations of SU(2)

You are already familiar with the fact that the spin of spin-$\frac{1}{2}$ fermions can be described using the Pauli matrices, i.e. they belong to a fundamental representation of SU(2). In this case, the eigenvectors of the T_z operator correspond to the "up" and "down" eigenstates, with eigenvalues of $s_z = \pm\frac{1}{2}$. Transitions between these eigenstates are obtained via linear combinations of the T_x and T_y generators, to form the usual raising and lowering operators:

$$T_\pm = T_x \pm iT_y. \tag{12}$$

Exercise 4: Re-express the result of Exercise 3 using T_z, $T_\pm$ and the 2×2 unit matrix.

We can construct objects of different spin by combining multiple spin-$\frac{1}{2}$ objects into larger mulitplets. For example, we could combine two spin-$\frac{1}{2}$ objects to form a state with a z component of -1, 0 or 1, illustrated on a simple number line as follows:

$s_{z1} = +1/2$		•	•
$s_{z1} = -1/2$	•	•	
s_z	-1	0	1

In group theory notation, this combination is written $\mathbf{2} \otimes \mathbf{2}$. In reality, the middle two states (with $s_z = 0$) are not eigenstates of the total spin s, instead we must rearrange the states into a spin triplet ($s = 1$) and a spin singlet ($s = 0$):

$s = 0$		•	
$s = 1$	•	•	•
s_z	-1	0	1

This is written as $\mathbf{3} \oplus \mathbf{1}$. The spin eigenstates with $s_z = 0$ can be written as

$$\begin{aligned} |\psi(s=0)\rangle &= \frac{1}{2}\left(|\psi_1(+1/2)\rangle|\psi_2(-1/2)\rangle - |\psi_1(-1/2)\rangle|\psi_2(+1/2)\rangle\right), \\ |\psi(s=1)\rangle &= \frac{1}{2}\left(|\psi_1(+1/2)\rangle|\psi_2(-1/2)\rangle + |\psi_1(-1/2)\rangle|\psi_2(+1/2)\rangle\right), \end{aligned} \quad (13)$$

where $|\psi_n(s_z)\rangle$ is the spin eigenstate for fermion n with the given s_z eigenvalue.

The singlet $s = 0$ wavefunction is not affected by the SU(2) group transformations – the generic spin rotation operator for this state is the identity matrix. The triplet state has three s_z eigenstates $(-1, 0, +1)$, and can be represented by a three-component vector. The generators of SU(2) rotations for this system are 3×3 matrices, like those used for SO(3) spatial rotations:

$$J_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (14)$$

These matrices obey the same group algebra as the matrices of Equation (10), but are clearly not equivalent. They form the adjoint representation of the SU(2) group, where the number of eigenstates equals the number of generators, in this case three. The physical interpretation of $J_{x,y,z}$ is identical to $T_{x,y,z}$ in the fundamental representation, including the definition of raising and lowering operators analogous to Equation (12).

Exercise 5: (Optional) An alternative definition of the adjoint representation is the set of matrices with elements given by

$$(J_a)_{bc} = -i f_{abc}. \tag{15}$$

Show that the matrices of Equation (14) can be obtained from this definition via the following transformations

$$\begin{aligned} J_x &= \frac{1}{\sqrt{2}} U (J_3 - J_1) U^{-1}, \\ J_y &= \frac{1}{\sqrt{2}} U (J_3 + J_1) U^{-1}, \\ J_z &= U J_2 U^{-1}, \end{aligned} \tag{16}$$

where

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & i \\ 0 & 1-i & 0 \\ i & 0 & 1 \end{pmatrix}. \tag{17}$$

We have seen how the adjoint representation can be related to the fundamental representation for SU(2). This relationship is rather generic: the adjoint representation can be obtained for any SU(N) from the group-theory product of two instances of the fundamental representation (in the above, $\mathbf{2} \otimes \mathbf{2}$). The $N^2 - 1$ non-trivial matrices will form the adjoint representation, while the identity transformation is associated with a group singlet.

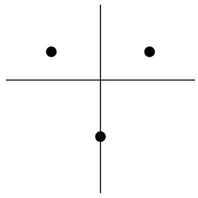
In the Standard Model, interacting fundamental matter particles (the fermions) belong to the fundamental representations of gauge groups.[2] Thus, there are two states of weak isospin (gauge group SU(2)), for example the electron and electron neutrino. Similarly, there are three colours of quark, corresponding to the fundamental representation of SU(3). The gauge bosons, on the other hand, belong to the adjoint representation for each group. Thus, there are three electroweak gauge bosons (corresponding, after electroweak symmetry breaking, to the W^+, Z^0 and W^- bosons), and eight ($= 3^2 - 1$) gluons.

4 SU(3) generators and baryonic systems

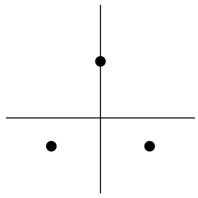
Much of the above discussion of SU(2) applies directly to SU(3), the symmetry associated with the strong nuclear force. Only the number of generators and the self-couplings described by the structure constants are different. We will use the concept of colour to explore the properties of SU(3), much as we used spin to explore SU(2).

[2] Non-interacting fermions are gauge group singlets. For example, electrons do not interact with the strong nuclear force, and are singlets of the corresponding gauge group.

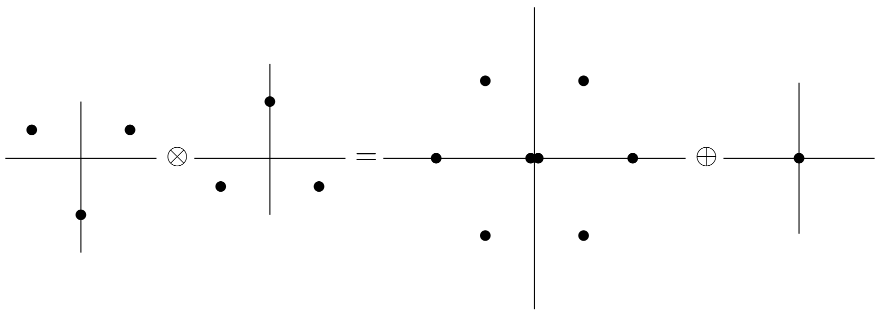
The fundamental representation of SU(3) (corresponding, e.g., to quark charges) has three elements, that can be illustrated on a two-dimensional plane, analagous to the line drawing of the SU(2) charges above:



In group theory notation, this is $\mathbf{3}$. In contrast to SU(2), there is also a fundamental $\bar{\mathbf{3}}$ representation, distinct from $\mathbf{3}$:



The adjoint representation can be obtained via the group product $\mathbf{3}\otimes\bar{\mathbf{3}}$. This produces a singlet state and an octet, in other words $\mathbf{3}\otimes\bar{\mathbf{3}} = \mathbf{8}\oplus\mathbf{1}$. The charges of these states can be seen by imagining the $\bar{\mathbf{3}}$ charges centered on each point of the $\mathbf{3}$ graph in turn. The resulting charge diagram is as follows:



Neglecting the singlet, there are therefore 8 members of the adjoint representation, 8 SU(3) generators and 8 types of gluon. From this picture, it can be easily understood that the gluon states around the edge of the octet correspond to the mixed colour states $r\bar{g}$, $g\bar{b}$ and so on. The two states at the origin are more complicated. The octet states, both orthogonal to the singlet state, can for example be written as

$$\frac{1}{\sqrt{2}}(r\bar{r} - g\bar{g}), \text{ and } \frac{1}{\sqrt{6}}(r\bar{r} + g\bar{g} - 2b\bar{b}). \tag{18}$$

The assignment of the colours red, green and blue to these states is clearly arbitrary.

These gluon states are related to the SU(3) generator matrices in the fundamental representation:[3]

$$
\begin{aligned}
T_1 &= \frac{1}{2}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad T_2 = \frac{1}{2}\begin{pmatrix}0 & -i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad T_3 = \frac{1}{2}\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix},\\
T_4 &= \frac{1}{2}\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}, \quad T_5 = \frac{1}{2}\begin{pmatrix}0 & 0 & -i\\ 0 & 0 & 0\\ i & 0 & 0\end{pmatrix},\\
T_6 &= \frac{1}{2}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}, \quad T_7 = \frac{1}{2}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -i\\ 0 & i & 0\end{pmatrix}, \quad T_8 = \frac{1}{2\sqrt{3}}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{pmatrix}.
\end{aligned} \tag{19}
$$

These (or rather the transform G defined in Equation (9)) act on 3×1 column vectors of the fundamental representation, i.e. quark states.

These matrices have a few other interesting properties. One is that T_1, T_2 and T_3 together look very similar to the SU(2) generator matrices. In fact, they satisfy all the SU(2) properties and themselves form a group, acting only on the first two colours. Thus SU(2) is actually a subgroup of SU(3).

Exercise 6: Compare the pairs (T_1, T_2), (T_4, T_5) and(T_6, T_7). Are other SU(2) subgroups of SU(3) possible? Can SU(3) be written as a product of SU(2) groups? Why/why not?

As with SU(2), the non-diagonal matrices are more usefully expressed as raising and lowering operators, capable of describing transitions between the three colour states. These operators are usually denoted $I^{\pm}$ (corresponding to the SU(2) subgroup just discussed, presumably named from the analogy with isospin), $V^{\pm}$ and $U^{\pm}$, defined as follows:

$$
\begin{aligned}
I^{\pm} &= T_1 \pm iT_2,\\
V^{\pm} &= T_4 \mp iT_5,\\
U^{\pm} &= T_6 \pm iT_7.
\end{aligned} \tag{20}
$$

Also note the relative sign change in the definition of $V^{\pm}$, this ensures that the raising operators operate in a circular fashion. More details are given in the lecture notes.

Exercise 7: The low-mass hadrons are also found to exhibit an approximate SU(3) symmetry, now understood to be due to the equivalence (with respect to the strong nuclear force) between the nearly massless up, down and strange quarks. Consider the elements of the baryon decu-

[3]The normalisation here is chosen such that $\mathrm{Tr}(T_a^2) = \frac{1}{2}$, compare Equation (11). The matrices used here are related to the λ_a matrices of the lecture notes by $T_a = \frac{1}{2}\lambda_a$.

plet, with the hypothesis that these are made up of constituent fermionic quarks. The baryon wavefunction must be antisymmetric with respect to the exchange of any two quarks. Show that this supports the hypothesis of quark colour by considering the exchange symmetry of the total baryonic wavefunction

$$\psi = \psi_F \psi_X \psi_S \psi_C, \tag{21}$$

where F stands for the quark flavour, X for the spatial wavefunction, S for spin and C for colour. Particular things to bear in mind include:

ψ_F : A flavour decouplet can be obtained from three constituent quarks via the following group-theoretic product:

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1} \tag{22}$$

if the quarks are described by the fundamental representation of SU(3) flavour.

ψ_X : It must be possible for all quarks to be in s-wave orbitals in the ground state, if the spin-$\frac{1}{2}$ octet is to exist.

ψ_S : The total spin of three spin-$\frac{1}{2}$ quarks would form some multiplet of the $\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2}$ group theory product. Experimentally, baryons in the decuplet have spin $s = \frac{3}{2}$.

ψ_C : All observed hadrons are colour singlets. You should obtain an expression for the $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$ colour singlet state, and from this deduce its properties under the exchange of two quarks. *Hint:* There is a useful analogy with the singlet state in Equation (13).