Tutorial 3: Gauge theories

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One of the key components of the Standard Model is the relationship between symmetries (of the Lagrangian density) and interactions between the particles. As we will see, the hints of this connection are already present in classical physics, but it is only in quantum mechanics that one can "derive" interactions from a symmetry principle. We will also consider non-Abelian symmetries, which have no classical counterpart, and finish with the problem of particle masses.

1 Gauge transformations and symmetry groups

1.1 Classical electromagnetism

The classical Lagrangian for an otherwise free particle of charge Q in an electromagnetic (EM) field is

$$L = \frac{1}{2}m\dot{\boldsymbol{x}}^2 + Q\dot{\boldsymbol{x}} \cdot \boldsymbol{A} - Q\Phi. \tag{1}$$

Exercise 1: Apply the appropriate Euler-Lagrange equation to Equation (1) and compare the result with Maxwell's equations to verify this assertion. *Hint:* Remember that $\boldsymbol{A}$ and Φ are functions of $\boldsymbol{x}$. A small amount of vector calculus is required.

Exercise 2: Calculate the canonical momentum and energy associated with Equation (1). You may assume the following definitions:

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{x}}}$$
$$\text{and } H = \boldsymbol{p} \cdot \dot{\boldsymbol{x}} - L, \tag{2}$$

where the Hamiltonian H describes the canonical energy.

In the last exercise, you should find that the canonical momentum and energy are not simply $m\dot{\boldsymbol{x}}$ and $\frac{1}{2}m\dot{\boldsymbol{x}}^2$. Instead, they have additional contributions from the electromagnetic potentials Φ and $\boldsymbol{A}$. Physically, these contributions can be understood as arising from the EM field itself, while maintaining that the *total* energy and momentum of the system (particle and field together) must be conserved. Formally, we can recover the particle's physical energy and momentum from the canonical variables by subtracting these extra contributions:

$$\begin{aligned} \boldsymbol{p}_{\text{phys.}} &= m\dot{\boldsymbol{x}} = \boldsymbol{p} - Q\boldsymbol{A}, \\ \text{and } E_{\text{phys.}} &= \frac{1}{2}m\dot{\boldsymbol{x}}^2 = H - Q\Phi. \end{aligned} \tag{3}$$

You will recall from your course lectures on electromagnetism that physical observables (including $\boldsymbol{p}_{\text{phys.}}$ and $E_{\text{phys.}}$) are unaffected by the following gauge transformation, where χ is an arbitrary function of space and time:

$$\Phi \to \Phi' = \Phi - \frac{\partial \chi}{\partial t}; \qquad \boldsymbol{A} \to \boldsymbol{A'} = \boldsymbol{A} + \nabla\chi. \tag{4}$$

Upon inspection of Equation (3), it is therefore clear that the canonical variables $\boldsymbol{p}$ and H *must* change under this transformation. In other words, the freedom to choose a gauge is intimately related to the non-trivial relationship between the canonical variables defined in Equation (2) and the physical observables.

These general principles apply equally to the Lagrangian density formulations of quantum mechanics and field theory. Naturally, classical variables must be replaced by appropriate quantum mechanical operators, but the form of the relationship in Equation (3) is identical. The point of this example is to remind you that the essential distinction between canonical variables and physical observables is present even in classical physics.

1.2 Electromagnetism in quantum mechanics

In the quantum mechanical context, canonical energy and momentum variables are replaced by differential operators in the usual way:

$$\hat{E} = i\frac{\partial}{\partial t} \text{ and } \hat{\boldsymbol{p}} = -i\nabla. \tag{5}$$

Or, in covariant notation:

$$\hat{p}_\mu = i\partial_\mu. \tag{6}$$

The equations of motion inevitably involve these operators acting on wavefunctions or fields, here referred to generically with the symbol ψ. In addition, covariant notation will be used for simplicity, as the separation into energy and momentum components (e.g. for application in the Schrödinger equation) is straightforward.

When electromagnetic interactions are included, physical four-momentum operators can be extracted by modifying the canonical operators in a way analagous to Equation (3):[1]

$$\begin{aligned}\hat{p}_\mu &= i\partial_\mu - QA_\mu \\ &= i(\partial_\mu + iQA_\mu) \\ &= i\mathcal{D}_\mu \end{aligned} \tag{7}$$

In the last line, the *covariant derivative* $\mathcal{D}_\mu = \partial_\mu + iQA_\mu$ is introduced, which replaces ∂_μ when calculating physical observables for interacting particles or fields.

One immediate issue is that the covariant derivative varies under a gauge transformation involving the field A^μ:

$$\begin{aligned} A_\mu \to A'_\mu &= A_\mu - \partial_\mu \chi \\ \Rightarrow \mathcal{D}_\mu \to \mathcal{D}'_\mu &= \partial_\mu + iQA_\mu - iQ\partial_\mu\chi. \end{aligned} \tag{8}$$

Thus, it would appear that the four-momentum expectation value $\langle\psi|\hat{p}^\mu|\psi\rangle$ changes, violating gauge invariance. This problem can be elegantly solved by additionally transforming the wavefunction ψ, which has no classical analogue, according to

$$\psi \to \psi' = e^{iQ\chi}\psi. \tag{9}$$

We note in passing that it alters an *internal* space of the wavefunction (its phase), rather than anything directly related to the external space-time.

> **Exercise 3:** Find out how the combination $\mathcal{D}_\mu\psi$ changes under the combined transformation of Equations (8) and (9). Use this result to show that $\langle\psi|\hat{p}^\mu|\psi\rangle$ is unaffected by the gauge transformation.

> **Exercise 4:** (Optional) Beginning with a free particle ($A^\mu = 0$), consider the gauge transformation given by $\chi = -\arg(\psi)/Q$. What are the resulting potentials and new wavefunction? Give a physical interpretation of the results in the case that ψ is a plane-wave function.

1.3 Relation to the U(1) symmetry group

It is well known that all observable quantites of any theory are invariant under a global phase transformation of a wave function or field

$$\psi \to \psi' = e^{i\phi}\psi, \tag{10}$$

so long as the phase angle ϕ does not depend on space-time coordinates. The prefactor $e^{i\phi}$ is a member of the U(1) group, discussed in the last

[1] From this point on, $\hat{p}^\mu$ is understood to represent the *physical* four-momentum, while $i\partial_\mu$ continues to describe the canonical four-momentum.

tutorial. The transformation in Equation (10) is referred to as a *global* U(1) transformation. When electromagnetic interactions are introduced, we find that now a much more stringent symmetry exists, that of the *local* U(1) transformation of Equation (9).

Thus, it appears that internal symmetries of a wavefunction or field are somehow related to their gauge interactions. Indeed, if we began with a non-interacting (free) particle, we could "derive" the electromagnetic interaction by requiring that the transformation in Equation (9) has no effect on observable quantities. We would then be forced to introduce a new field A^μ associated with the four-momentum operator, which would transform as given in Equation (8).

All gauge interactions of the Standard Model are ultimately derived in this way. Starting from the Lie group operators applied to an internal space (called a Hilbert space), one demands that physically observable quantities are unchanged upon arbitrary local rotations within this space. Additional complications arise, however, when the group in question is a non-Abelian group. It is this topic that we will consider next.

2 Non-Abelian groups in gauge theories

Recall from the previous tutorial that a non-Abelian group is one where at least one structure constant f_{abc} is non-zero, where the structure constants are defined via

$$[T_a, T_b] = T_a T_b - T_b T_a = i f_{abc} T_c. \tag{11}$$

For SU(2), the structure constants are equal to the completely antisymmetric Levi-Civita symbol ϵ_{abc}. In the case of SU(3), the constants are more complicated, and given in the lecture notes.

We will start by generalising the transformations derived in the previous section, without assuming that the group elements commute. In general, a state ψ will transform as follows:

$$\psi \to \psi' = G\psi = e^{i\boldsymbol{\alpha}\cdot\boldsymbol{T}}\psi, \tag{12}$$

where G is a local Hilbert space transformation. We introduce a covariant derivative to absorb changes to the Lagrangian density resulting from this transformation:

$$\mathcal{D}_\mu = \partial_\mu + igW_\mu = \partial_\mu + ig\boldsymbol{T}\cdot\boldsymbol{W}_\mu. \tag{13}$$

Here, $\boldsymbol{W}_\mu$ is a vector of new gauge fields, with a size corresponding to the number of group generators, while g is an associated coupling strength, analogous to the electric charge. Under the gauge transformation, the field W_μ transforms as follows:

$$W_\mu \to W'_\mu = GW_\mu G^{-1} + \frac{i}{g}(\partial_\mu G)G^{-1}. \tag{14}$$

Exercise 5: Show that under this transformation $\mathcal{D}'_\mu \psi' = G\mathcal{D}_\mu \psi$.

Exercise 6: Show that Equation (14) reduces to Equation (8) for the U(1) transformation of Equation (9) if $g = Q$.

Alternatively, we can also write Equation (14) in terms of the component fields

$$\begin{aligned}\boldsymbol{T}\cdot\boldsymbol{W}'_\mu &= e^{i\boldsymbol{\alpha}\cdot\boldsymbol{T}}\boldsymbol{T}\cdot\boldsymbol{W}_\mu e^{-i\boldsymbol{\alpha}\cdot\boldsymbol{T}} + \frac{i}{g}\left(\partial_\mu e^{i\boldsymbol{\alpha}\cdot\boldsymbol{T}}\right)e^{-i\boldsymbol{\alpha}\cdot\boldsymbol{T}},\\ \text{or}\quad T_a W'_{\mu,a} &= e^{i\alpha_b T_b} T_a W_{\mu,a} e^{-i\alpha_c T_c} + \frac{i}{g}\left(\partial_\mu e^{i\alpha_b T_b}\right)e^{-i\alpha_c T_c}, \qquad (15)\end{aligned}$$

where the Einstein summation convention for indices has been assumed.

The above expressions describe a generic, arbitrarily large, transformation of a gauge field derived through symmetry principles. For our puposes it is useful to also consider the perturbative regime, when the transformation G is close to the identity. In this case, $|\boldsymbol{\alpha}|$ is small, and $e^{i\boldsymbol{\alpha}\cdot\boldsymbol{T}} \approx 1 + i\boldsymbol{\alpha}\cdot\boldsymbol{T}$. If we neglect second-order terms in $\boldsymbol{\alpha}$ and $\partial_\mu\boldsymbol{\alpha}$, we can write Equation (15) as

$$\begin{aligned}T_a W'_{\mu,a} &\simeq (1 + i\alpha_b T_b) T_a W_{\mu,a}(1 - i\alpha_c T_c) - \frac{1}{g}(\partial_\mu \alpha_a) T_a (1 - i\alpha_b T_b)\\ &= T_a W_{\mu,a} - i(\alpha_c T_a T_c - \alpha_b T_b T_a) W_{\mu,a} - \frac{1}{g}(\partial_\mu\alpha_a) T_a + \mathcal{O}(\boldsymbol{\alpha}^2) \qquad (16)\end{aligned}$$

Here, we note that the indices b and c on the right hand side are arbitrary, so we can rewrite $\alpha_c T_a T_c$ as $\alpha_b T_a T_b$ to obtain

$$\begin{aligned}T_a W'_{\mu,a} &= T_a W_{\mu,a} - i\alpha_b [T_a, T_b] W_{\mu,a} - \frac{1}{g}(\partial_\mu \alpha_a) T_a\\ &= T_a W_{\mu,a} + f_{abc}\alpha_b T_c W_{\mu,a} - \frac{1}{g}(\partial_\mu\alpha_a) T_a \qquad (17)\end{aligned}$$

This suggests the following solution:

$$W'_{\mu,a} = W_{\mu,a} + f_{abc}\alpha_c W_{\mu,b} - \frac{1}{g}(\partial_\mu \alpha_a). \qquad (18)$$

Exercise 7: Show explicitly that $\mathcal{D}'_\mu\psi' = G\mathcal{D}_\mu\psi$ for the infinitecimal transformation of Equation (18), to first order in $|\boldsymbol{\alpha}|$.

3 Building an interacting Lagrangian

Consider the Lagrangian density for a free fermionic field ψ:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi. \qquad (19)$$

As previously discussed, a gauge interaction for this fermion may be introduced by replacing ∂_μ by the covariant derivative from Equation (13):

$$\begin{aligned}\mathcal{L} &= \bar{\psi}(i\gamma^\mu \mathcal{D}_\mu - m)\psi \\ &= \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - g\bar{\psi}\gamma^\mu \boldsymbol{T}\cdot\boldsymbol{W}_\mu\psi. \end{aligned} \tag{20}$$

The final term of Equation (20) represents interactions between the fermion and the gauge field. These modify the propagation of the free fermion field, described in the first two terms. However, we must also consider the possibility of Lagrangian density terms that involve only the fields $\boldsymbol{W}_\mu$. It turns out that the only Lorentz-covariant object allowed by gauge symmetry that we can form from the gauge fields alone is the commutator of the covariant derivative, $[\mathcal{D}_\mu, \mathcal{D}_\nu]$. This can be evaluated as follows:

$$\begin{aligned}[\mathcal{D}_\mu, \mathcal{D}_\nu] &= [\partial_\mu + ig\boldsymbol{T}\cdot\boldsymbol{W}_\mu, \partial_\nu + ig\boldsymbol{T}\cdot\boldsymbol{W}_\nu] \\ &= [\partial_\mu, \partial_\nu] + ig[\partial_\mu, \boldsymbol{T}\cdot\boldsymbol{W}_\nu] + ig[\boldsymbol{T}\cdot\boldsymbol{W}_\mu, \partial_\nu] \\ &\quad - g^2[\boldsymbol{T}\cdot\boldsymbol{W}_\mu, \boldsymbol{T}\cdot\boldsymbol{W}_\nu]. \end{aligned} \tag{21}$$

The first commutator is evidently zero. The commutator in the second term can be found by considering what happens when this operates on a wavefunction:

$$\begin{aligned}[\partial_\mu, \boldsymbol{T}\cdot\boldsymbol{W}_\nu]\psi &= \partial_\mu(\boldsymbol{T}\cdot\boldsymbol{W}_\nu\psi) - \boldsymbol{T}\cdot\boldsymbol{W}_\nu(\partial_\mu\psi) \\ &= (\partial_\mu\boldsymbol{T}\cdot\boldsymbol{W}_\nu)\psi. \end{aligned} \tag{22}$$

Note that the end result does not depend in any way on the wavefunction we temporarily introduced. Similarly, $[\boldsymbol{T}\cdot\boldsymbol{W}_\mu, \partial_\nu] = -\partial_\nu\boldsymbol{T}\cdot\boldsymbol{W}_\mu$. The final term of Equation (21) is evaluated using the group algebra, giving the following result

$$\begin{aligned}[\mathcal{D}_\mu, \mathcal{D}_\nu] &= ig\,\boldsymbol{T}\cdot(\partial_\mu\boldsymbol{W}_\nu - \partial_\nu\boldsymbol{W}_\mu) - ig^2 f_{abc}T_a W_{\mu,b}W_{\nu,c} \\ &= T_a[ig(\partial_\mu W_{\nu,a} - \partial_\nu W_{\mu,a}) - ig^2 f_{abc}W_{\mu,b}W_{\nu,c}]. \end{aligned} \tag{23}$$

In the U(1) case, the structure constants vanish, and Equation (23) is proportional to the field tensor $F_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu$, familiar from electromagnetism. In the non-Abelian case, the first terms of Equation (23) also describe free fields that propagate in the vacuum much like the photon, but the final term will not vanish.

The only Lorentz scalar field propagation term that we can construct using $F_{\mu\nu} \propto [\mathcal{D}_\mu, \mathcal{D}_\nu]$ is $F_{\mu\nu}F^{\mu\nu}$. The final term of Equation (23) will yield terms proportional to $(\partial_\mu W_\nu)W^\mu W^\nu$ and $W_\mu W_\nu W^\mu W^\nu$, where the group generator indices have been suppressed. These terms ultimately correspond to interactions between the various components of the $\boldsymbol{W}_\mu$ field, something that we will return to later in the course.

4 Gauge theories and particle masses

While gauge theory looks on the surface to be an elegant way to describe natural forces, it suffers from one important problem: it requires all (non-singlet) particles to be massless, in order to work.

In the case of gauge bosons, to give a mass to the $\boldsymbol{W}^\mu$ field would require a term in the Lagrangian density proportional to $\boldsymbol{W}^\mu \cdot \boldsymbol{W}_\mu$ (recall the Proca equation from tutorial 1). However, the terms available in Equation (23) do not allow any quadratic terms the Lagrangian density, because these would break the gauge symmetry. This is a problem when it comes to describing the weak nuclear force, as the W and Z bosons that mediate it have substantial masses of about 80 and 90 GeV, respectively.

What is perhaps less obvious is that, due to the peculiar nature of the weak interaction, all weakly interacting fermions must also be massless. It might naively be thought that a fermionic mass term $-m\bar{\psi}\psi$ (c.f. Equation (19)) would always remain invariant under any transformation $\psi \to G\psi$. It turns out that this is not the case for the weak nuclear force, which acts differently on the left- and right-handed components of ψ. Recalling the first tutorial, we can rewrite ψ as a sum of these components:

$$\begin{aligned} \psi &= \psi_\mathrm{L} + \psi_\mathrm{R} \\ &= \frac{1}{2}(1-\gamma^5)\psi + \frac{1}{2}(1+\gamma^5)\psi. \end{aligned} \tag{24}$$

The left- and right-handed spinors transform differently under the SU(2) symmetry transformation for the weak force. Specifically, ψ_L is an SU(2) doublet, while ψ_R is a singlet, and so invariance of the fermionic mass terms cannot be assumed.

Exercise 8: Evaluate the conjugate field $\overline{\psi_\mathrm{L}}$ in terms of $\bar{\psi}$ and γ^5.[a] Use this result, together with the equivalent expression for $\overline{\psi_\mathrm{R}}$, to evaluate the fermion mass term $m\bar{\psi}\psi$ in terms of the chiral eigenstates.

[a]The long bar in $\overline{\psi_\mathrm{L}}$ indicates that the chirality projection operator is applied before conjugation.

It should be clear from the previous exercise that the fermion mass terms are not allowed by the weak force SU(2) symmetry. We will return to the puzzle of particle masses in the SM in the next tutorial.