Tutorial 5: Calculating cross sections and Feynman rules
With answers

Dr. M Flowerdew

December 9, 2015

1 Preamble

Now that we have all the interactions of the Standard Model defined, we want to know how to use them to make predictions of the rates and properties of interactions between the different fields. In this tutorial, we will calculate the cross-section for one particular process using relativistic quantum mechanics, as an illustration of how this can be done. Along the way, we will verify that applying the Feyman rules for the corresponding diagram would yield the same result.

A generic particle physics experiment is illustrated in Figure 1. This shows the initial state ψ_{in} being transformed into the final state ψ_{out} by some interaction within an experimental apparatus. For simplicity, we will assume that ψ_{in} and ψ_{out} can be modeled using plane-wave momentum eigenstates, which only interact within a limited space-time volume VT. Outside that volume, all interactions are turned off.

Exercise 1: Why is it necessary to restrict the interaction volume when using plane waves? *Hint:* Consider how the assumption of plane waves breaks down in realistic experiments for $x, t \to \pm\infty$.

Answer: Plane waves ($\psi \propto \exp(-ik_\mu x^\mu)$) are a convenient approximation, but cannot describe all aspects of a particle collision experiment. In real life, the interacting particles will be spatially and temporally constrained — for example to within the volume of two colliding beams, and only while the experiment is operational. Two perfectly plane waves could, however, interact at any space and time — the probability does not fall to zero in the limit of $x, t \to \pm\infty$. Allowing the plane waves to interact only within a limited space-time volume is a mathematically simple way to resolve this problem. Other approaches exist, for example one could treat the incoming particles as wave packets, by introducing a small spread to their momentum distributions, but this would be more mathematically complicated to describe.

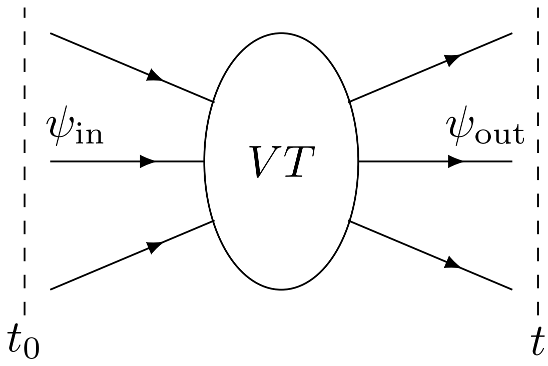

Figure 1: Schematic diagram of "in" and "out" states considered in high energy particle collisions.

These states ψ_{in} and ψ_{out} are defined on boundary regions at times t_0 and t, respectively — eventually, we can imagine taking these to negative and positive infinity. The states propagate freely according to some Hamiltonian H_0, so that each wavefunction at any time t can be formally expressed as[1]

$$\psi(t) = e^{-iH_0(t-t_0)}\psi(t_0). \tag{1}$$

This conventional approach to the evolution of wavefunctions is also called the *Schrödinger Picture*. For our purposes, it is however useful to use the *Interaction Picture* (or Dirac Picture) instead. In the Interaction Picture, the plane-wave time evolution is factored into the quantum mechanical operators, which allows changes due to interactions to be seen more clearly.

The Interaction Picture state is defined as follows:

$$\hat{\psi}(t) = e^{iH_0(t-t_0)}\psi(t). \tag{2}$$

Outside the interaction volume, these states therefore remain constant in time, and only evolve within the interaction volume VT.

Inside the volume, the Hamiltonian now has an extra term, H_1, describing the interaction(s). The Schrödinger Picture wavefunction evolves according to

$$i\frac{\partial\psi}{\partial t} = H_0\psi + H_1\psi. \tag{3}$$

[1]For simplicity, we will assume that both ψ_{in} and ψ_{out} evolve according to H_0. In general they may not, for example when the collision is inelastic. This requires a more careful treatment of which basis states we use, but the essential points of the following arguments remain similar.

Exercise 2: Use the Dirac equation, expressed in the form of Equation (3), to show that the interaction Hamiltonian for fermion in an electromagnetic field is $H_1 = q\gamma^0\gamma^\mu A_\mu(x)$.

Answer: We insert the EM interaction into the Dirac equation by using the covariant derivative as usual:

$$\begin{aligned} (i\gamma^\mu \mathcal{D}_\mu - m)\,\psi = 0 \\ \Rightarrow (i\gamma^\mu\partial_\mu - q\gamma^\mu A_\mu - m)\,\psi = 0 \end{aligned}$$

To express this in the form of Equation (3), we need to expand the sum of partial derivatives, noting (perhaps surprisingly) that $\gamma^\mu\partial_\mu = \gamma^0\partial_0 + \gamma^1\partial_1 + \gamma^2\partial_2 + \gamma^3\partial_3$:

$$\begin{aligned} \Rightarrow i\gamma^0\partial_0\psi &= \left(-i\gamma^1\partial_1 - i\gamma^2\partial_2 - i\gamma^3\partial_3 + m\right)\psi + q\gamma^\mu A_\mu\psi \\ \text{So } i\partial_0\psi &= \left(-i\gamma^0\gamma^1\partial_1 - i\gamma^0\gamma^2\partial_2 - i\gamma^0\gamma^3\partial_3 + \gamma^0 m\right)\psi + q\gamma^0\gamma^\mu A_\mu\psi. \end{aligned}$$

The term in parentheses on the right-hand side is clearly present even if the fermion is free, we therefore associate this with H_0. The remaining term is $H_1\psi$, and has the required form.

Exercise 3: Use Equation (3) to show that the equation of motion for $\hat{\psi}$ is

$$i\frac{\partial\hat{\psi}}{\partial t} = \hat{H}_1\hat{\psi}, \tag{4}$$

where

$$\hat{H}_1 = e^{iH_0(t-t_0)}H_1 e^{-iH_0(t-t_0)}. \tag{5}$$

Answer: To do this, we simply compute the time derivative of $\hat{\psi}$:

$$\begin{aligned} i\frac{\partial\hat{\psi}}{\partial t} &= i\frac{\partial}{\partial t}\left(e^{iH_0(t-t_0)}\psi(t)\right) \\ &= i\left(\frac{\partial}{\partial t}e^{iH_0(t-t_0)}\right)\psi + ie^{iH_0(t-t_0)}\frac{\partial\psi}{\partial t} \\ &= -H_0 e^{iH_0(t-t_0)}\psi + e^{iH_0(t-t_0)}\left(H_0\psi + H_1\psi\right). \end{aligned}$$

The first two terms cancel, as H_0 trivially commutes with $e^{iH_0(t-t_0)}$. The third term then becomes our final result:

$$\begin{aligned} i\frac{\partial\hat{\psi}}{\partial t} &= e^{iH_0(t-t_0)}H_1\psi = e^{iH_0(t-t_0)}H_1 e^{-iH_0(t-t_0)}e^{iH_0(t-t_0)}\psi \\ &= \hat{H}_1\hat{\psi}. \end{aligned}$$

1.1 Perturbation theory using the Interaction Picture

With Equation (4), we can formally solve for $\hat{\psi}(t)$ within the interaction volume, assuming that H_1 is "small", such that we may use perturbation theory:

$$\hat{\psi}(t) = \hat{\psi}_{\text{in}}(t_0) - i \int_{t_0}^{t} \hat{H}_1(t_1)\hat{\psi}_{\text{in}}(t_1)\,\mathrm{d}t_1. \tag{6}$$

We can use this solution to replace $\hat{\psi}_{\text{in}}$ within the integral, and we see the beginnings of a perturbative expansion in $\hat{H}_1$:

$$\begin{aligned}\hat{\psi}(t) &= \hat{\psi}_{\text{in}}(t_0) - i \int_{t_0}^{t} \hat{H}_1(t_1) \left\{ \hat{\psi}_{\text{in}}(t_0) - i \int_{t_0}^{t_1} \hat{H}_1(t_2)\hat{\psi}_{\text{in}}(t_2)\,\mathrm{d}t_2 \right\} \mathrm{d}t_1 \\ &= \hat{\psi}_{\text{in}}(t_0) - i \int_{t_0}^{t} \hat{H}_1(t_1)\hat{\psi}_{\text{in}}(t_0)\,\mathrm{d}t_1 \\ &\quad + (-i)^2 \int_{t_0}^{t}\int_{t_0}^{t_1} \hat{H}_1(t_1)\hat{H}_1(t_2)\hat{\psi}_{\text{in}}(t_0)\,\mathrm{d}t_2\,\mathrm{d}t_1 + \mathcal{O}(\hat{H}_1^3).\end{aligned} \tag{7}$$

Inverting Equations (2) and (5) and rearranging the subscripts in the double integral, we can obtain the expansion for the Schrödinger Picture wavefunction $\psi(t)$:

$$\begin{aligned}\psi(t) = {}& e^{-iH_0(t-t_0)}\psi_{\text{in}}(t_0) && \text{(8a)}\\ &+ \int_{t_0}^{t} e^{-iH_0(t-t_1)}\left(-iH_1(t_1)\right) e^{-iH_0(t_1-t_0)}\psi_{\text{in}}(t_0)\,\mathrm{d}t_1 && \text{(8b)}\\ &+ \int_{t_0}^{t}\int_{t_0}^{t_2} e^{-iH_0(t-t_2)}\left(-iH_1(t_2)\right) e^{-iH_0(t_2-t_1)} \\ &\qquad \times \left(-iH_1(t_1)\right) e^{-iH_0(t_1-t_0)}\psi_{\text{in}}(t_0)\,\mathrm{d}t_1\,\mathrm{d}t_2 && \text{(8c)}\\ &+ \mathcal{O}(H_1^3).\end{aligned}$$

These terms are understood in the following way, noting that H_1 and H_0 do not necessarily commute:

(8a) Evolution of the unperturbed state to time t, i.e. no interaction.

(8b) Evolution to time t_1, at which point an interaction occurs, after which the resulting state evolves freely to time t.

(8c) Two interactions occur at times t_1 and t_2, with free propagation in between those times.

Using the result of Equation (8), we can calculate the amplitude for a transition to any particular final state $\psi_{\text{out}}(t)$ by calculating the overlap integral $\langle\psi_{\text{out}}(t)|\psi(t)\rangle$ in the usual way. This we will proceed to do for one simple example.

2 Fermion-fermion scattering cross section

In this section, we compute the interaction cross section for the process shown in Figure 2. The two scattering fermions are assumed to be of different flavours, for example an electron and a muon, to avoid complications of multiple diagrams contributing to the same final state.

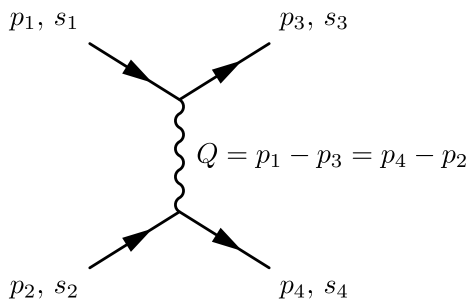

Figure 2: Diagram of first-order electromagnetic scattering of two non-identical fermions.

The four fermion legs are characterised by their four-momentum p_i and spin s_i. Neglecting normalisation factors for now, the initial and final state wavefunctions (in the Schrödinger Picture) are

$$\begin{aligned} \psi_1(t_0) &= u_{s_1}(p_1)e^{-i(E_1 t_0 - \boldsymbol{p_1}\cdot\boldsymbol{x})}, \qquad \psi_2(t_0) = u_{s_2}(p_2)e^{-i(E_2 t_0 - \boldsymbol{p_2}\cdot\boldsymbol{x})}, \\ \psi_3(t) &= u_{s_3}(p_3)e^{-i(E_3 t - \boldsymbol{p_3}\cdot\boldsymbol{x})}, \qquad \psi_4(t) = u_{s_4}(p_4)e^{-i(E_4 t - \boldsymbol{p_4}\cdot\boldsymbol{x})}. \end{aligned} \tag{9}$$

The $u_s(p)$ factors are Dirac spinors. Note that they are already in the correct form for the Interaction Picture, i.e. with free-particle propagation removed. In much of the following, these will be written as u_1, u_2 etc., for simplicity.

Following the rules outlined in the previous section (in particular, Equation (8)), we can write down an expression for the amplitude of this transition, from the perspective of particle 1:

$$A = \int_{t_0}^{t} \langle\psi_3(t)|e^{-iH_0(t-t_1)}\left(-iH_1(t_1)\right)e^{-iH_0(t_1-t_0)}|\psi_1(t_0)\rangle\,\mathrm{d}t_1 \tag{10}$$

The exponentials involving H_0 act on the final and initial states to give $e^{-iE_3(t-t_1)}$ and $e^{-iE_1(t_1-t_0)}$, respectively. We will also substitute the appropriate form for H_1 for an electromagnetic interaction to give

$$\begin{aligned} A &= \int_{t_0}^{t}\int_V u_3^\dagger e^{i(E_3 t - \boldsymbol{p_3}\cdot\boldsymbol{x})}e^{-iE_3(t-t_1)}(-i)q_1\gamma^0\gamma^\mu A_\mu(\boldsymbol{x},t_1) \\ &\qquad\qquad \times e^{-iE_1(t_1-t_0)}e^{-i(E_1 t_0 - \boldsymbol{p_1}\cdot\boldsymbol{x})}u_1\,\mathrm{d}^3\boldsymbol{x}\,\mathrm{d}t_1 \\ &= -iq_1\int_{t_0}^{t}\int_V u_3^\dagger\gamma^0\gamma^\mu u_1 A_\mu(\boldsymbol{x},t_1)e^{-i[(E_1-E_3)t_1 - (\boldsymbol{p_1}-\boldsymbol{p_3})\cdot\boldsymbol{x}]}\,\mathrm{d}^3\boldsymbol{x}\,\mathrm{d}t_1 \end{aligned}$$

$$= -iq_1\bar{u}_3\gamma^\mu u_1 \int_{VT} A_\mu(x_1)e^{-i(p_1-p_3)x_1}\,\mathrm{d}^4x_1. \tag{11}$$

The Dirac spinors do not depend on space-time coordinates and have therefore been removed from the integral. The factor $q_1\bar{u}_3\gamma^\mu u_1$ is interpreted as the fermion current for particle 1, such that the amplitude is proportional to (the integral of) $j_1^\mu A_\mu$.

The integral over A_μ is called the *propagator* for the virtual boson exchanged (in this case) between the fermions. In the limit where VT encompasses all of space-time, this integral becomes the Fourier transform of A_μ. We will see that the factor $(p_1 - p_3)$ in the exponential is related to the conservation of four-momentum. There are no propagators for the fermions (which have external lines in the Feynman diagram), as the phase factors associated with these have already been absorbed into the initial and final state vectors.

2.1 The photon propagator and Green's functions

In this section, we will evaluate the photon propagator part of Equation (11).

$$\tilde{A}^\mu(p_1 - p_3) = \int_{VT} A^\mu(x_1)e^{-i(p_1-p_3)x_1}\,\mathrm{d}^4x_1. \tag{12}$$

This can be found by using the appropriate Green's function. The Green's function $G(x, x')$ associated with a differential operator $\mathcal{O}$ is defined as the function which satisfies the following equation

$$\mathcal{O}G(x, x') = \delta(x - x'). \tag{13}$$

This Green's function can be used to solve for any field satisfying the equation $\mathcal{O}\psi(x) = f(x)$, with the result $\psi(x) = \int G(x, x')f(x')\,\mathrm{d}x'$.

One interesting property of Equation (13) is that the Fourier transform of the delta function is constant, which we chose to be 1 by the form of Equation (12). This means that the Fourier transform of the Green's function is just the inverse of $\mathcal{O}$, when $\mathcal{O}$ is expressed in momentum space variables. Or, more precisely:

$$\mathcal{O}\tilde{G}(p) = 1. \tag{14}$$

Exercise 4: Show that, for a fermion with $\mathcal{O} = i\gamma^\mu\partial_\mu - m = \not{p} - m$, that

$$\tilde{G}(p) = \frac{(\not{p} + m)}{p^2 - m^2}. \tag{15}$$

Hint: You cannot divide by $\not{p}$. The form of the answer contains a hint of how to resolve this issue.

Answer: We start with a more complete proof of Equation (14).

$$\begin{aligned}\mathcal{O}\tilde{G}(p) &= \int \mathcal{O}G(x,x')e^{-ip(x-x')}\,\mathrm{d}^4x' \\ &= \int \delta(x-x')e^{-ip(x-x')}\,\mathrm{d}^4x' \\ &= e^0 = 1.\end{aligned}$$

In the case that $\mathcal{O} = \not{p} - m$, we can solve the problem by premultiplying Equation (14) by $\not{p} + m$:

$$\begin{aligned}(\not{p}+m)(\not{p}-m)\tilde{G}(p) &= (\not{p}+m) \\ \text{or } \tilde{G}(p) &= \frac{(\not{p}+m)}{p^2-m^2}.\end{aligned}$$

In our example, the photon field satisfies the equation $\Box A^\mu = j_2^\mu$, where j_2^μ is the current from particle 2. In this case, the operator $\mathcal{O}$ is the D'Alembertian $\Box = \partial^\mu\partial_\mu$, or Q^2 in momentum space if Q is the photon four-momentum. Therefore, the integral in Equation (12) may be written

$$\tilde{A}^\mu(Q) = \frac{1}{Q^2}\tilde{j}_2^\mu. \tag{16}$$

Exercise 5: Compute $\tilde{A}^\mu(Q)$. You will need to express the current from particle 2 in the Schrödinger Picture, and evaluate the resulting Fourier Transform integral. Show that the result can be expressed as the product of the particle 2 current in the Interaction Picture, the photon propagator $g^{\mu\nu}/Q^2$, and a delta function that ensures overall four-momentum conservation.

Answer: The Interaction Picture current j_2^μ is, by analogy with Equation (11), $q_2\bar{u}_4\gamma^\mu u_2$. We take the form of the Fourier integral from Equation (12), and allowing for the extra phase factors from Equation (9), we find that

$$\begin{aligned}\tilde{A}^\mu(Q) &= \frac{1}{Q^2}\int_{VT} q_2\left(\bar{u}_4 e^{ip_4x_1}\right)\gamma^\mu\left(u_2 e^{-ip_2x_1}\right)e^{-i(p_1-p_3)x_1}\,\mathrm{d}^4x_1 \\ &= \frac{g^{\mu\nu}}{Q^2}\int_{VT} q_2\bar{u}_4\gamma_\nu u_2 e^{-i(p_1+p_2-p_3-p_4)x_1}\,\mathrm{d}^4x_1 \\ &= \frac{g^{\mu\nu}}{Q^2}q_2\bar{u}_4\gamma_\nu u_2(2\pi)^4\delta^4(p_1+p_2-p_3-p_4).\end{aligned}$$

The first factor is the photon propagator, which is followed by the (Interaction Picture) fermion current and finally a delta function that ensures that $p_1 + p_2 = p_3 + p_4$.

Substituting the result of the above exercise into Equation (11) gives the complete amplitude for this process:

$$A = -(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\bar{u}_3(-iq_1\gamma^\mu)u_1\left(\frac{-ig_{\mu\nu}}{Q^2}\right)\bar{u}_4(-iq_2\gamma^\nu)u_2. \quad (17)$$

The inserted factors of i follow the standard Feynman rules. While irrelevant in this simple example, they are important when interference effects between multiple diagrams are considered. The factor

$$i\mathcal{M}_{fi} = \bar{u}_3(-iq_1\gamma^\mu)u_1\left(\frac{-ig_{\mu\nu}}{Q^2}\right)\bar{u}_4(-iq_2\gamma^\nu)u_2 \quad (18)$$

is the *matrix element* for the $i \to f$ transition.

Exercise 6: Re-derive the matrix element by using the Feynman rules given in the lecture notes. You may neglect the factors of $(2\pi)^4\delta^4(\ldots)$ from the fermion-photon vertices, as this is already taken into account in the value of Q and the delta function in Equation (17).

Answer: The Feynman rule for external fermions states that we should begin with an outgoing fermion and trace it through to the corresponding incoming fermion. In the case of outgoing fermion 3, this gives us the following factors:

$$\begin{aligned}\text{Outgoing fermion 3:} &\quad \bar{u}_3,\\ \text{1–3–}\gamma\text{ vertex:} &\quad -iq_1\gamma^\nu,\\ \text{Incoming fermion 1:} &\quad u_1.\end{aligned}$$

This precisely matches the first fermion current term in Equation (18). Next, the rule for the photon propagator is:

$$\text{Photon propagator:} \quad \frac{-ig_{\mu\nu}}{Q^2+i\epsilon}.$$

This agrees with Equation (18), except for the $i\epsilon$ term in the denominator, which inserted to facilitate the integration of complex numbers. We are not using that approach, and so in our case the limit of $\epsilon \to 0$ is appropriate.

Finally, we consider the outgoing fermion 4:

$$\begin{aligned}\text{Outgoing fermion 4:} &\quad \bar{u}_4,\\ \text{2–4–}\gamma\text{ vertex:} &\quad -iq_2\gamma^\nu,\\ \text{Incoming fermion 2:} &\quad u_2.\end{aligned}$$

2.2 Squaring the amplitude

The cross-section will, in the end, depend upon the square of the amplitude just derived. For now, we will simply focus on the matrix element part of this calculation. From Equation (18):

$$
\begin{aligned}
|M_{fi}|^2 &= \left| \bar{u}_3(-iq_1\gamma^\mu)u_1 \left(\frac{-i}{Q^2}\right) \bar{u}_4(-iq_2\gamma_\mu)u_2 \right|^2 \\
&= \frac{q_1^2 q_2^2}{Q^4}(\bar{u}_3\gamma^\mu u_1)(\bar{u}_4\gamma_\mu u_2)(\bar{u}_2\gamma_\nu u_4)(\bar{u}_1\gamma^\nu u_3) \\
&= \frac{q_1^2 q_2^2}{Q^4} L_1^{\mu\nu} L_{2\mu\nu} \qquad (19)
\end{aligned}
$$

where

$$
L_1^{\mu\nu} = \bar{u}_3\gamma^\mu u_1 \bar{u}_1\gamma^\nu u_3 \qquad (20)
$$

and similarly for $L_{2\mu\nu}$.

In the end, we wish to average over initial spin states and sum over final spin states. There are four initial spin states, as each incoming fermion has two possible helicities. Therefore, the matrix element we want to calculate is actually $\frac{1}{4}\sum_{\text{spins}}|M_{fi}|^2$. This can be evaluated using standard gamma matrix algebra, in particular using the following result (reinstating spin and momentum labels):

$$
\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m. \qquad (21)
$$

Exercise 7: Show that

$$
\sum_{\text{spins}} L_1^{\mu\nu} = 4\left[p_1^\mu p_3^\nu - g^{\mu\nu} p_1\cdot p_3 + p_1^\nu p_3^\mu + m_1^2 g^{\mu\nu}\right]. \qquad (22)
$$

The fermion tensor $L_{2\mu\nu}$ has the same form. *Hint:* First write out the complete expression for the lepton tensor, including all spinor indices.

Answer: Using α, β, ρ and σ as spinor indices, we proceed by using some standard gamma-matrix tricks:

$$
\begin{aligned}
\sum_{\text{spins}} L_1^{\mu\nu} &= \sum_{s_1,s_3} \bar{u}^\alpha_{s_3}(p_3)\gamma^{\mu,\alpha\beta}u^\beta_{s_1}(p_1)\bar{u}^\rho_{s_1}(p_1)\gamma^{\nu,\rho\sigma}u^\sigma_{s_3}(p_3) \\
&= \gamma^{\mu,\alpha\beta}\left[\not{p}_1 + m_1\right]^{\beta\rho}\gamma^{\nu,\rho\sigma}\left[\not{p}_3 + m_1\right]^{\sigma\alpha} \\
&= \mathrm{Tr}\left(\gamma^\mu(\not{p}_1 + m_1)\gamma^\nu(\not{p}_3 + m_1)\right) \\
&= \mathrm{Tr}\left(\gamma^\mu\not{p}_1\gamma^\nu\not{p}_3\right) + m_1^2\,\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\right) \\
&= 4\left[p_1^\mu p_3^\nu - g^{\mu\nu} p_1\cdot p_3 + p_1^\nu p_3^\mu + m_1^2 g^{\mu\nu}\right].
\end{aligned}
$$

Exercise 8: Combine Equations (19) and (22) to show that

$$\frac{1}{4}\sum_{\text{spins}}|M_{fi}|^2 = 4\frac{q_1^2q_2^2}{Q^4}\left[2(p_1\cdot p_2)(p_3\cdot p_4)+2(p_1\cdot p_4)(p_2\cdot p_3)\right.$$
$$\left.-2m_1^2(p_2\cdot p_4)-2m_2^2(p_1\cdot p_3)+4m_1^2m_2^2\right] \quad (23)$$

Answer: First we just substitute for the lepton tensors in Equation (19):

$$\frac{1}{4}\sum_{\text{spins}}|M_{fi}|^2 = \frac{16q_1^2q_2^2}{4Q^4}\left[p_1^\mu p_3^\nu - g^{\mu\nu}p_1\cdot p_3 + p_1^\nu p_3^\mu + m_1^2g^{\mu\nu}\right]$$
$$\times\left[p_{2\mu}p_{4\nu} - g_{\mu\nu}p_2\cdot p_4 + p_{2\nu}p_{4\mu} + m_2^2g_{\mu\nu}\right].$$

We then expand the product. This is done explicitly below, with each row corresponding to one term in the first bracket above.

$$\frac{1}{4}\sum_{\text{spins}}|M_{fi}|^2 = \frac{4q_1^2q_2^2}{Q^4}\times$$
$$\{(p_1\cdot p_2)(p_3\cdot p_4)-(p_1\cdot p_3)(p_2\cdot p_4)+(p_1\cdot p_4)(p_2\cdot p_3)+m_2^2(p_1\cdot p_3)$$
$$-(p_1\cdot p_3)(p_2\cdot p_4)+4(p_1\cdot p_3)(p_2\cdot p_4)-(p_1\cdot p_3)(p_2\cdot p_4)-4m_2^2(p_1\cdot p_3)$$
$$+(p_1\cdot p_4)(p_2\cdot p_3)-(p_1\cdot p_3)(p_2\cdot p_4)+(p_1\cdot p_2)(p_3\cdot p_4)+m_2^2(p_1\cdot p_3)$$
$$+m_1^2(p_2\cdot p_4)-4m_1^2(p_2\cdot p_4)+m_1^2(p_2\cdot p_4)+4m_1^2m_2^2\}$$

All of the terms proportional to $(p_1\cdot p_3)(p_2\cdot p_4)$ cancel, and upon collecting everything else together we obtain Equation (23).

Finally, we introduce the Mandelstam variables, so that this result may be written in a more compact form:

$$\begin{aligned} s &= (p_1+p_2)^2 = (p_3+p_4)^2 \approx 2p_1\cdot p_2 \approx 2p_3\cdot p_4 \\ t &= (p_1-p_3)^2 = (p_2-p_4)^2 \approx -2p_1\cdot p_3 \approx -2p_2\cdot p_4 \\ u &= (p_1-p_4)^2 = (p_2-p_3)^2 \approx -2p_1\cdot p_4 \approx -2p_2\cdot p_3 \end{aligned} \quad (24)$$

The dot-product approximations are valid if all incoming and outgoing particles are highly relativistic. In this limit, Equation (23) becomes

$$\frac{1}{4}\sum_{\text{spins}}|M_{fi}|^2 = 2\frac{q_1^2q_2^2}{t^2}\left(s^2+u^2\right) \quad (25)$$

2.3 Fermi's Golden Rule

Now we have squared the matrix element, we need to consider what other information is required to calculate an interaction cross section. The full rate is given by Fermi's Golden Rule

$$\mathrm{d}\sigma = \frac{\frac{1}{VT}|A|^2\rho(E_f)\,\mathrm{d}E_f}{F}, \tag{26}$$

where $\rho(E_f)$ is the density of final states with respect to the final energy $\mathrm{d}E_f$, division by VT normalises the interaction volume, and F is the particle flux.

Consider first the normalised square of the amplitude A. This will result in the product of two delta functions, which is not formally defined. However, we can proceed by noting that the second delta function will evaluate to $\delta^4(0)$ and will contribute a factor VT to the final result when integrated over space-time. In other words

$$\begin{aligned}\frac{1}{VT}|A|^2 &= \frac{1}{VT}(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\int_{VT} e^{-i\cdot 0}\,\mathrm{d}^4x|\mathcal{M}_{fi}|^2 \\ &= (2\pi)^4\delta^4(p_1+p_2-p_3-p_4)|\mathcal{M}_{fi}|^2. \end{aligned} \tag{27}$$

The Lorentz invariant phase space factor $\rho(E_f)\,\mathrm{d}E_f$ is defined as

$$\rho(E_f)\,\mathrm{d}E_f = \frac{1}{2E_3}\frac{1}{2E_4}\frac{\mathrm{d}^3\boldsymbol{p_3}}{(2\pi)^3}\frac{\mathrm{d}^3\boldsymbol{p_4}}{(2\pi)^3}. \tag{28}$$

Exercise 9: Evaluate $\delta^4(p_1+p_2-p_3-p_4)\rho(E_f)\,\mathrm{d}E_f$ for highly relativistic particles in the centre-of-momentum frame. Use the delta functions to eliminate $\mathrm{d}^3\boldsymbol{p_4}$ and $\mathrm{d}p_3$, leaving only the dependence on the solid angle $\mathrm{d}\Omega$.

Answer: We wish to evaluate

$$\delta^4(p_1+p_2-p_3-p_4)\rho(E_f)\,\mathrm{d}E_f = \frac{1}{4(2\pi)^6}\delta^4(p_1+p_2-p_3-p_4)\frac{\mathrm{d}^3\boldsymbol{p_3}\mathrm{d}^3\boldsymbol{p_4}}{E_3E_4}$$

First, we note that the integral over $\mathrm{d}^3\boldsymbol{p_4}$ is trivial. Only the delta function depends on $\boldsymbol{p_4}$, and this only serves to ensure that E_4 has the value we expect for an on-shell particle. In other words:

$$\int \delta^3(\boldsymbol{p_1}+\boldsymbol{p_2}-\boldsymbol{p_3}-\boldsymbol{p_4})\,\mathrm{d}^3\boldsymbol{p_4} = 1.$$

This leaves the integral over $\mathrm{d}^3\boldsymbol{p_3}$, which can be written as $p_3^2\,\mathrm{d}p_3\,\mathrm{d}\Omega$.

Then, we note that $\frac{\mathrm{d}(E_3+E_4)}{E_3+E_4} = \frac{p_3 \mathrm{d}p_3}{E_3 E_4}$, so our integral becomes

$$\frac{1}{4(2\pi)^6}\delta(E_1+E_2-E_3-E_4)\frac{p_3^2\,\mathrm{d}p_3\,\mathrm{d}\Omega}{E_3E_4} = \frac{1}{4(2\pi)^6}\delta(E_1+E_2-E_3-E_4)\frac{p_3\,\mathrm{d}(E_3+E_4)}{E_3+E_4}\,\mathrm{d}\Omega.$$

We can then integrate over $\mathrm{d}(E_3+E_4)$ using the delta function, to give us our final answer:

$$\int_{E_3+E_4}\frac{1}{4(2\pi)^6}\delta(E_1+E_2-E_3-E_4)\frac{p_3\,\mathrm{d}(E_3+E_4)}{E_3+E_4}\,\mathrm{d}\Omega = \frac{1}{4(2\pi)^6}\frac{p_3}{E_1+E_2}\,\mathrm{d}\Omega \approx \frac{1}{8(2\pi)^6}\,\mathrm{d}\Omega.$$

In the last part, we have assumed that the particles are highly relativistic, so that $p_3 \approx E_3 = E_1 \approx E_2$.

Finally, there is the particle flux. This is the product of the particle densities (normalised to $2E$ particles per unit volume) and the difference of their velocities.

Exercise 10: Show that $F = 2s$ in the relativistic limit. *Hint:* Consider a unit volume of colliding particles.

Answer: We start from the basic definition of the particle flux,

$$F = n_1 n_2 |\delta \boldsymbol{v}|,$$

where n_i is the number of particles of type i and $\delta\boldsymbol{v}$ is the difference of their velocities. If we consider a unit volume, then $n_i = 2E_i$, and $\delta\boldsymbol{v} = 2$ for highly relativistic particles in the centre-of-momentum frame (recall that $c = 1$). Therefore:

$$F = 8E_1E_2.$$

Relating this to the Mandelstam variables, we note that

$$s = 2p_1p_2 = 2(E_1E_2 - \boldsymbol{p_1}\cdot\boldsymbol{p_2}) = 2[E_1E_2 - (-E_1E_2)] = 4E_1E_2.$$

It immediately follows that $F = 2s$.

Putting this together, we obtain the differential cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s}|\mathcal{M}_{fi}|^2. \qquad (29)$$

Substituting in the result from Equation (25), we obtain our final result in the highly-relativistic limit:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{q_1^2 q_2^2}{32\pi^2 s} \frac{s^2 + u^2}{t^2}. \tag{30}$$