# Tutorial 6: The parton model of hadrons

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## 1 Fermion-fermion scattering: recap

In this tutorial, we will discuss the nature of hadrons, and some of the experimental evidence for partons (quarks and gluons) inside the proton and neutron. These experiments usually focus on interactions of composite hadrons (or nuclei) with elementary leptons, in so-called *deep inelastic scattering*, or DIS. Some other evidence for the existence of quarks was already discussed in Tutorial 2. For lack of time, we will only consider electromagnetic interactions, however the same methods can be used to probe the charged- and neutral-current weak interactions of nuclei.

We begin by considering interactions between elementary fermions, before discussing the parton model proper. We finished the last tutorial with the differential cross section for elastic $t$-channel EM scattering of two dissimilar fermions in the centre of momentum (c.m.) frame. The diagram for this process is redrawn in Figure 1 using symbols conventionally used when describing DIS. The cross section in the c.m. frame (denoted by hats) is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\hat{\Omega}} = \frac{q_1^2 q_2^2}{32\pi^2 s}\frac{s^2 + u^2}{t^2}, \tag{1}$$

in the limit of massless fermions. The right hand side of the equation is fully Lorentz invariant, however the left side is not.

> **Exercise 1:** Consider the Mandelstam variable $t$, defined in the previous tutorial. Express $t$ in terms of $s$ and the angle of deflection $\hat{\theta}$. By considering the effect of an integration over $\hat{\phi}$, show that $\mathrm{d}\hat{\Omega}$ can be replaced by $\frac{4\pi}{s}\mathrm{d}(-t)$.

When considering DIS, differential cross-sections are usually rewritten

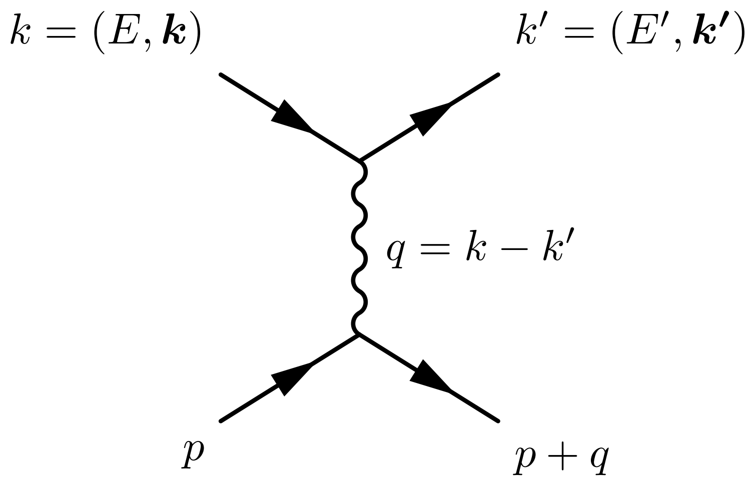



Figure 1: Diagram of first-order electromagnetic scattering of two non-identical fermions.

in terms of these four variables:

$$
\begin{aligned}
&\text{C.M. energy} && s = 2k \cdot p \\
&\text{Momentum transfer} && Q^2 = -q^2 = 2k \cdot k' = -t \\
&\text{Inelasticity} && y = \frac{2p \cdot q}{s} = 1 + \frac{u}{s} \\
&\text{Björken scaling variable} && x = \frac{k \cdot k'}{p \cdot q}
\end{aligned} \tag{2}
$$

Note that these variables are not independent. Given any three, the fourth can be calculated via

$$Q^2 = sxy. \tag{3}$$

> **Exercise 2:** If all of the particles in Figure 1 are fundamental, then the scattering process is elastic and $(p+q)^2 = 0$ (or $(p+q)^2 = p^2$ in the massive case). What does this imply for $x$? How many independent degrees of freedom are there for elastic scattering?

In what follows, we will also normalise the charges to $e$, writing $q_i^2 = Q_i^2 e^2 = 4\pi Q_i^2 \alpha$. Expressed using these variables, Equation (1) becomes

$$
\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2} &= Q_1^2 Q_2^2 \frac{2\pi\alpha^2}{Q^4} \left[1 + (1-y)^2\right] \\
&= Q_1^2 Q_2^2 \frac{2\pi\alpha^2}{Q^4} \left[y^2 + 2(1-y)\right]
\end{aligned} \tag{4}
$$

## 2 Deep inelastic scattering

Next, we consider inelastic scattering of a fermion (e.g. an electron) from a hadron (e.g. a proton). Hadrons are composite particles, and therefore the interaction does not have the same simple form as Equation (4). To see how it might differ, recall that the matrix element for Figure 1 could be written as the product of two fermion tensors, evaluated independently for each interacting particle:

$$|M_{fi}|^2 = \frac{q_1^2 q_2^2}{Q^2} L_1^{\mu\nu} L_{2\mu\nu}. \tag{5}$$

For the hadron case, we simply replace one fermion tensor with an unknown hadron tensor $W_{\mu\nu}$:

$$|M_{fi}|^2 = \frac{q_1^2 q_2^2}{Q^2} L^{\mu\nu} W_{\mu\nu}. \tag{6}$$

The hadron tensor is constrained by the same symmetries as the lepton tensor, but with arbitrary non-perturbative coefficients. The fact that the collision is inelastic introduces an additional degree of freedom, such that the Björken scaling variable $x$ is no longer constant (compare Exercise 2). The differential cross section can be parameterised in terms of arbitrary *structure functions* for a hadron $A$, $F_i^A(x, Q^2)$, so that Equation (4) becomes[1]

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x\,\mathrm{d}Q^2} = Q_1^2 Q_2^2 \frac{4\pi\alpha^2}{xQ^4} \left[y^2 x F_1^A(x, Q^2) + (1-y) F_2^A(x, Q^2)\right]. \tag{7}$$

Equation (7) would need to be extended in the case of weak interactions, as some terms allowed in the weak matrix element are forbidden to electromagnetism. In all cases the structure functions must be determined experimentally, although the parton model introduced next makes predictions about some of their properties.

### 2.1 The parton model and structure function predictions

We consider now the process illustrated in Figure 2. Here, a fundamental fermion (in practice this must be a lepton) collides with a composite hadron. The hadron is assumed to be composed of multiple *partons*, only one of which scatters with the other fermion. The colliding parton is assumed to carry a fraction $\xi$ of the hadron's four-momentum.[2] We assume that the partons are themselves fermionic (i.e. quarks), and that the scattering is elastic at the parton level.

[1] Some of the coefficient factors have changed with respect to Equation (4). This is purely conventional and has no real significance.

[2] The parton's momentum perpendicular to the hadron's motion is zero. Strictly speaking, this assumption is valid only in the so-called *infinite momentum frame*, and in general where any partonic transverse momentum may be neglected relative to $p$.

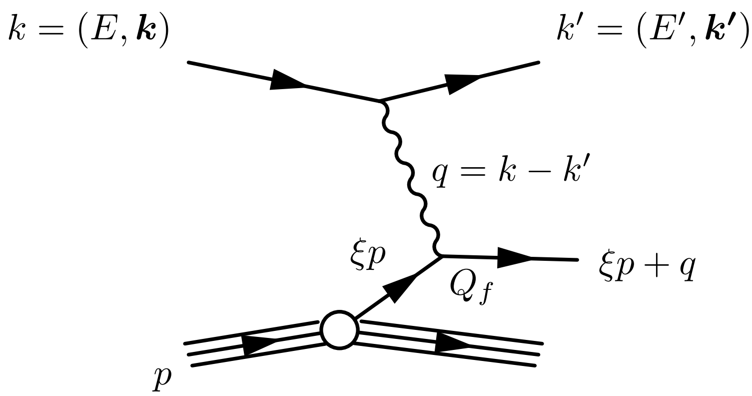



Figure 2: Diagram of deep inelastic scattering of a lepton and a hadron.

If the lepton-parton collision is elastic, then the squared mass of the colliding parton is unchanged by the collision. Therefore

$$\begin{aligned}
&(\xi p)^2 = (\xi p + q)^2 \\
&\Rightarrow 0 = 2\xi p \cdot q + q^2 \\
&\Rightarrow \xi = \frac{-q^2}{2p \cdot q} = \frac{Q^2}{sy} = x.
\end{aligned} \tag{8}$$

Thus, the Björken variable $x$ is identified as the momentum fraction carried by the interacting parton, and we have no more use for $\xi$.

Any hadron will contain a variety of different parton flavours, which we must sum over to find the total structure function. The probability of finding a parton $a$ with a momentum $xp$ in hadron $A$ is described by a *parton density function* (pdf) $f_A^a(x)$.[3] The product $xf_A^a(x)$ is a momentum probability density, and is therefore subject to normalisation:

$$\sum_{a \in A} \int_0^1 x f_A^a(x) \, \mathrm{d}x = 1. \tag{9}$$

In addition, if there are $n_q$ valence quarks of type $q$, then it is expected that

$$\int_0^1 f_A^q(x) - f_A^{\bar{q}} \, \mathrm{d}x = n_q. \tag{10}$$

Using these pdfs and Equation (4), we can write the observed double-

[3]In general, $a$ will be used to refer to *any* parton, including anti-quarks for example.

differential cross section for the process in Figure 2 as

$$\begin{aligned}\frac{\mathrm{d}\sigma(x,Q^2)}{\mathrm{d}x\,\mathrm{d}Q^2} &= \sum_a f_A^a(x)\frac{\mathrm{d}\sigma_a(Q^2)}{\mathrm{d}Q^2}\\ &= Q_1^2\frac{2\pi\alpha^2}{xQ^4}\left[y^2+2(1-y)\right]x\sum_a Q_a^2 f_A^a(x).\end{aligned}\tag{11}$$

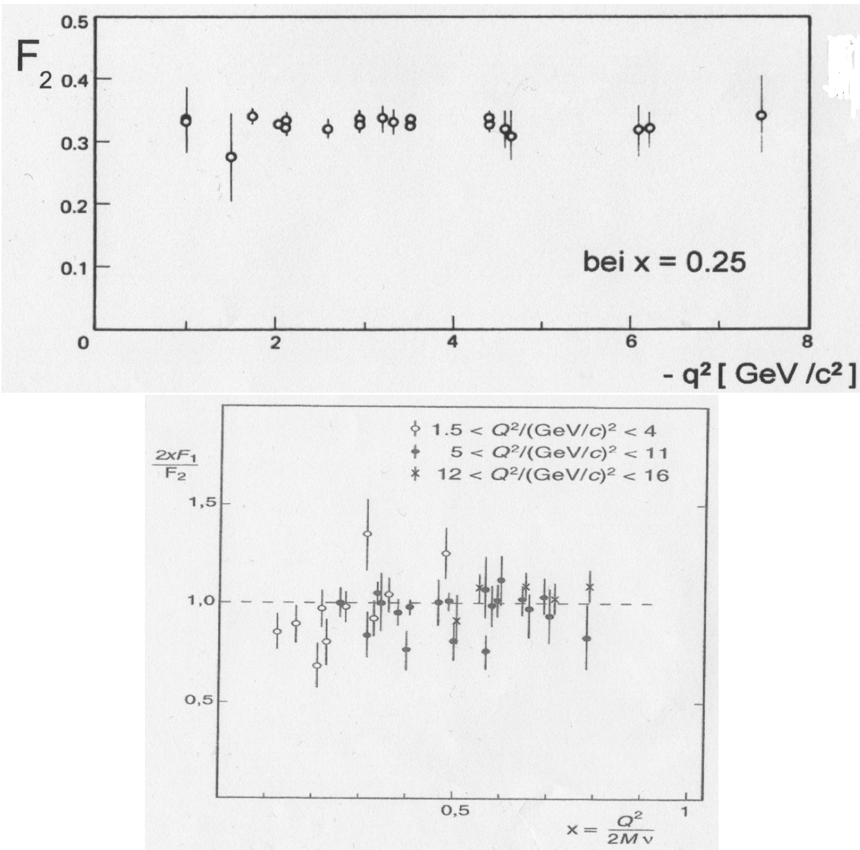



Figure 3: Top: Illustration of Björken scaling, where $F_2^p(x=0.25)$ is constant over a wide range of $Q^2$ values. Bottom: Illustration of the Callan-Gross relation as a function of $x$ for three $Q^2$ ranges.

Comparing with Equation (7), we deduce the following:

$$\begin{aligned}F_1^A(x,Q^2) &= \frac{1}{2}\sum_a Q_a^2 f_A^a(x)\\ F_2^A(x,Q^2) &= x\sum_a Q_a^2 f_A^a(x),\end{aligned}\tag{12}$$

which leads to three important observations:

**Björken scaling:** $F_1^A$ and $F_2^A$ depend only on $x$, and not on the energy scale $Q^2$. This contrasts strongly with, say, resonance production, where the cross section varies strongly with $Q^2$.

**The Callan-Gross relation:** $F_2^A(x) = 2xF_1^A(x)$, a relationship which depends on the partons' spin.[4]

**The differential cross-section** is proportional to $[1 + (1 - y^2)]F_2^A$ (compare Equation (4)). For this reason, $F_1^A(x)$ is seldom used nowadays, and deviations from Equation (11) are described by terms proportional to $[1 - (1 - y)^2]xF_3^A$ and $y^2 F_\mathrm{L}^A$.

Both of the first two properties were confirmed at SLAC, as shown for protons in Figure 3. These, and other observations, led to the adoption of the quark parton model.

> **Exercise 3:** Draw a diagram analogous to Figure 2 for the process $p\bar{p} \to ZX$, where $X$ refers to an unspecified collection of hadrons. What (anti-)quark flavour combinations should be considered when computing the cross-section for this process? Use this to argue that the $Z$ boson production cross-section in $p\bar{p}$ collisions is given approximately by
>
> $$\sigma_{p\bar{p}\to ZX} = \sum_q \int_0^1 \int_0^1 f_p^q(x_p) f_{\bar{p}}^{\bar{q}}(x_{\bar{p}}) \sigma_{q\bar{q}\to Z}(\hat{s})\, \mathrm{d}x_p\, \mathrm{d}x_{\bar{p}}, \tag{13}$$
>
> where $\sqrt{\hat{s}}$ is the partonic centre-of-mass energy.

# 3 The gluon and scaling violation

Suppose we say that, for a fixed value of $Q^2$, we define $u(x)$ and $\bar{u}(x)$ as the up and anti-up pdfs of the proton, respectively, and similarly define $d(x)$ and $\bar{d}(x)$ for the down quark.[5] Equation (12) then predicts the value of $F_2^p$ in terms of these pdfs:

$$F_2^p(x) = \frac{4}{9}x\{u(x) + \bar{u}(x)\} + \frac{1}{9}x\{d(x) + \bar{d}(x)\}. \tag{14}$$

The neutron is related to the proton by an isopin transformation that swaps $u$ and $d$ quarks. Assuming that the isospin symmetry is exact, we can therefore write $F_2^n$ in terms of the proton pdfs:

$$F_2^n(x) = \frac{1}{9}x\{u(x) + \bar{u}(x)\} + \frac{4}{9}x\{d(x) + \bar{d}(x)\}. \tag{15}$$

[4]Scalar partons would satisfy $F_1^A = 0$, for example.

[5]The antiquark densities are included for generality, but could be neglected for this discussion. See the end of this section if their inclusion is confusing.

Experimentally, the integrated $F_2^{p/n}$ values are found from low-energy (compared to the LHC) DIS with protons and deuterons:

$$\int_0^1 F_2^p(x)\,\mathrm{d}x = 0.18 \quad \text{and} \quad \int_0^1 F_2^n(x)\,\mathrm{d}x = 0.12. \tag{16}$$

**Exercise 4:** Use Equations (14) to (16) to show that

$$\int_0^1 x\{u(x)+\bar{u}(x)\}\,\mathrm{d}x = 0.36 \quad \text{and} \quad \int_0^1 x\{d(x)+\bar{d}(x)\}\,\mathrm{d}x = 0.18. \tag{17}$$

Comment on the apparent discrepancy with respect to Equation (9).

The previous exercise shows that much of the proton's momentum cannot be accounted for by the quarks. Another feature of DIS that cannot be explained using quarks alone is shown in Figure 4. This illustrates evolution of the $e^+p$ cross-section as $x$ and $Q^2$ are varied. For $x \sim 0.1$ the effect of Björken scaling can be seen, as in Figure 3. The cross-section is not constant for the largest values of $x$ due to $Z$ boson exchange, which we neglected for simplicity. However, for $x \lesssim 0.01$ the cross-section increases markedly with $Q^2$ in a way that cannot be attributed to weak interactions, an effect called *scaling violation*.

Both the "missing" momentum of the proton and the low-$x$ scaling violation can be explained by the gluon. Gluons are invisible to electroweak scattering experiments, which is why their momentum is not included in the calculation of Exercise 4. Scaling violation can be understood as a higher-order correction to the simple picture of static partons. As a gluon propagates, we know that it can spontaneously create a virtual quark-antiquark pair. Normally, they quickly annihilate, but the effect gives rise to *sea quarks*, i.e. the probability of a collision with an anti-up (or anti-down) quark is non-zero. It is also possible to produce strange, charm and even bottom quarks in this manner. The gluon splitting rate is, however, highly dependent on both $x$ and $Q^2$, and this breaks Björken scaling. As a result, all DIS quantities, including the pdfs, are generally quoted as a function of both $x$ and $Q^2$.

**Exercise 5:** Many DIS experiments are performed on so-called *isoscalar* targets, which have an equal number of protons and neutrons (e.g. $^{12}$C). In this case, we can define a *nucleon* structure function for charged lepton interactions

$$F_2^N = \frac{1}{2}\left(F_2^p + F_2^n\right). \tag{18}$$

Evaluate $F_2^N$ assuming only contributions from $u$, $d$, $\bar{u}$ and $\bar{d}$ partons.

A similar structure function, $F_2^{\nu N}$ can be defined for charged-current

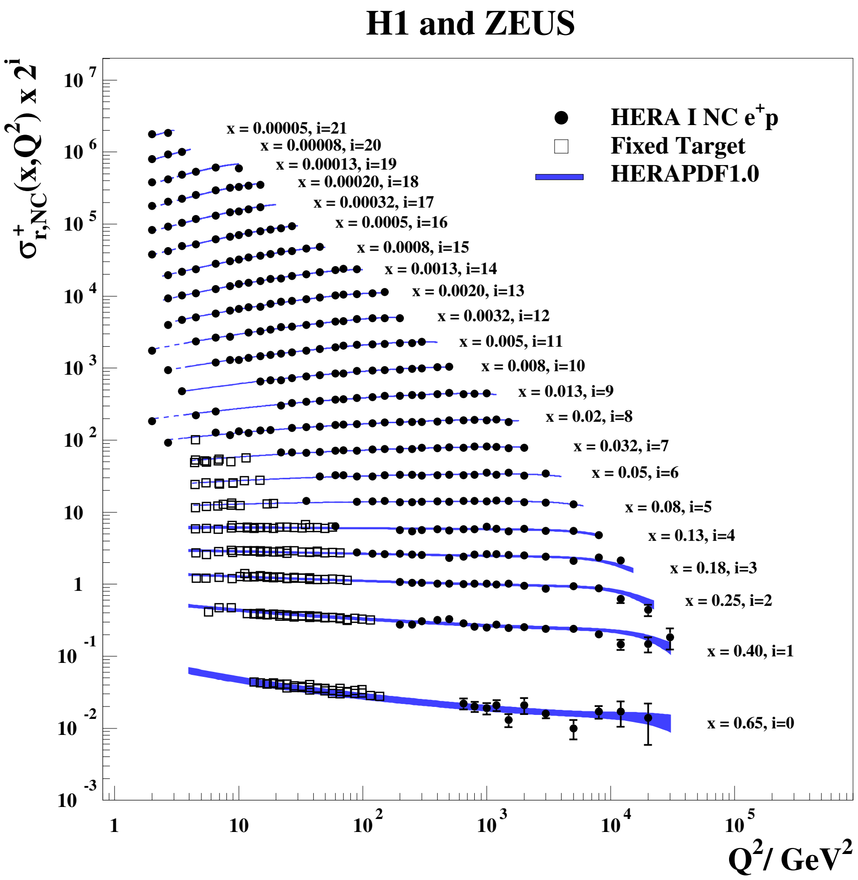



Figure 4: Reduced cross-section (i.e. with certain coefficients removed) for $e^+p$ collisions as a function of $Q^2$ measured at the HERA collider for selected values of $x$. Each data series is offset from the last by a factor of two, for clarity. The reduced cross-section is defined so that it is roughly equal to $F_2^p(x, Q^2)$ across much of the parameter space.

neutrino interactions.[a] The conventional definitions of $F_2^{\nu p}$ and $F_2^{\nu n}$ have the same form as Equation (12), except that $Q_a^2$ is replaced by 2 if a charged-current $\nu a$ interaction is possible, or 0 if it is not possible. Evaluate $F_2^{\nu N}$, taking care over which quark flavours contribute to the sum. Finally, evaluate the ratio $F_2^N / F_2^{\nu N}$.

[a] In this case, $\nu$ refers strictly to neutrinos, and not to anti-neutrinos.

# 4 Parton density functions

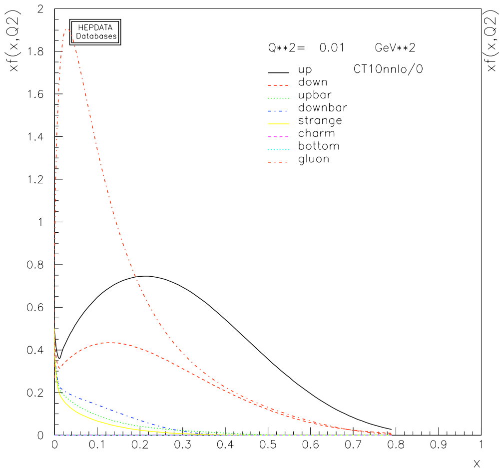
In the previous section, we saw how inclusive measurements of the structure functions can give hints as to the structure of hadrons. Combinations of more detailed differential analyses allow the individual pdfs to be calculated. We will focus on the most highly-studied hadron, the proton. Figure 5 shows measured parton density functions for three values of $Q^2$, spanning six orders of magnitude from a value much smaller than $\Lambda_{\text{QCD}}^2$ to the electroweak scale $\mathcal{O}(100\text{ GeV})^2$.

**Exercise 6:** What is the valence quark content of the proton? Consider a naive model for $N$ non-interacting quarks where the valence quark pdfs are proportional to $\delta(x - \frac{1}{N})$. Compare the up- and down-quark pdfs in Figure 5a to this model. Qualitatively explain a) the shape of the pdfs with respect to the naive model and b) their relative normalisation.

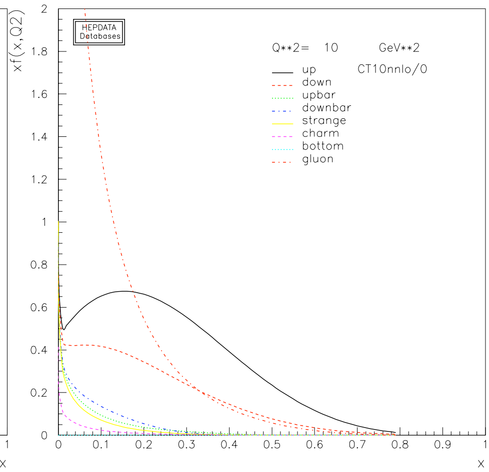
Comparing Figures 5a and 5b, we see that the valence quark pdfs do not change substantially at high $x$ as $Q^2$ increases. This is consistent with Björken scaling. Scaling violation becomes more evident at low values of $x$, for the reasons discussed in Section 3. The gluon pdf is dominant for $x \lesssim 0.2$, and also the sea quark pdfs are concentrated at low $x$.

In the lowest panel of Figure 5, a much higher value of $Q^2$ is shown, comparable to $m_Z^2$. Now the proton's momentum is dominated by gluons and sea quarks at low $x$. To balance this out, the valence pdfs at high $x$ decrease slightly. The dominance of sea quarks in this final panel has important implications for modern hadron collider design.

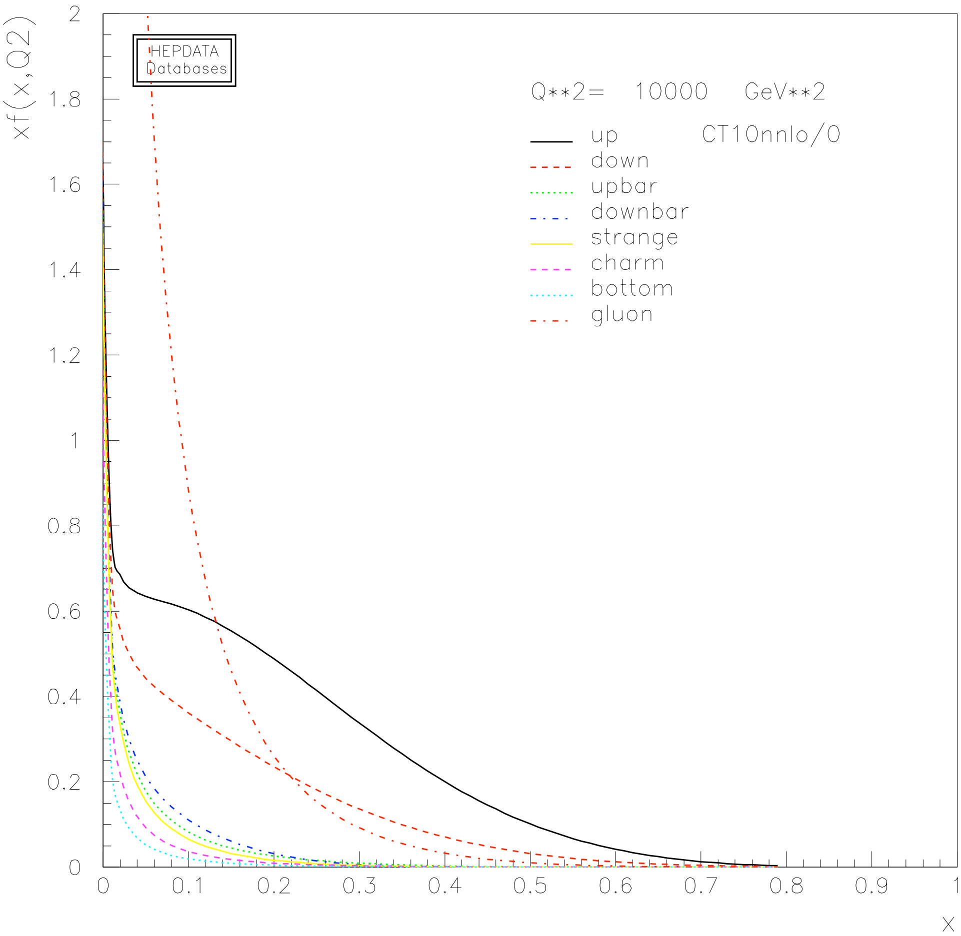
**Exercise 7:** Consider $Z$ boson production in $p\bar{p}$ collisions, as discussed in Exercise 3. The Tevatron collided protons and anti-protons with $\sqrt{s} \approx 2$ TeV, while the production of an on-shell $Z$ boson requires $\sqrt{\hat{s}} = m_Z \sim 100$ GeV. What value of $x$ will the colliding (anti-)quarks typically have in this interaction? You may assume for simplicity that the two quarks have equal energies (i.e. that $x_1 = x_2 = x$). Using Figure 5c and Equation (13), discuss the relative rate of $Z$ boson production under these conditions and in $pp$ collisions at the same value of $\sqrt{s}$. *Hint:* Assume that $f_p^q = f_{\bar{p}}^{\bar{q}}$ — why is this a valid assumption?




(a) $Q^2 = 0.01$ GeV$^2$




(b) $Q^2 = 10$ GeV$^2$




(c) $Q^2 = 10^4$ GeV$^2$

Figure 5: Parton density functions from a recent next-to-next-to-leading order fit to data from multiple experiments at three different $Q^2$ values.

**Exercise 8:** Repeat the previous calculation for the LHC, assuming that $\sqrt{s} = 14$ TeV. Compare again (as much as is possible) with Figure 5c, and discuss the relative advantages of $pp$ and $p\bar{p}$ collisions at this energy.