Tutorial 1

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"Tests des Standardmodells der Teilchenphysik"

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Lagrangian Formalism

The fundamental quantity of classical mechanics is the action, S, the time integral of the Lagrangian, L. In order to describe the evolution of the system of mass-point particles, it is customary to use the *generalized coordinates*, q, and *generalized velocities*, $\dot{q}$.

1. How are the Lagrangian function L density and action S exactly defined?

When describing the evolution of systems from one given configuration to another between times t_0 and t_1, the Hamilton's principle, often referred to as the *principle of least action*, states that the system chooses the path for which S is an extremum (usually a minimum).

2. Derive the Euler-Lagrange equation of motion based on this principle, that is show the equation of motion for the system $\{q(t),\, \dot{q}(t)\}$ can take the following known form

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \; . \tag{1}$$

Analogously, in a local field theory the Lagrangian can be expressed as the spatial integral of the Lagrangian density, $\mathcal{L} = \mathcal{L}(\varphi_i, \varphi_\mu)$, by substituting $q \to \varphi(x)$ and $\dot{q} \to \dot{\varphi}(x)$

$$\mathcal{L} = \mathcal{L}\left(\varphi,\, \partial_\mu \varphi\right) \tag{2}$$

where

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0},\, \boldsymbol{\nabla}\right) \quad \text{and} \quad \partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial x^0},\, -\boldsymbol{\nabla}\right) \tag{3}$$

is the 4-gradient,

$$x^\mu = (x^0,\, \mathbf{x}), \quad x_\mu = g_{\mu\nu} x^\nu = (x^0,\, -\mathbf{x}) \tag{4}$$

and

$$g_{\nu\mu} = g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad g^{\mu\nu} g_{\mu\nu} = \delta^\mu{}_\nu = 4 \tag{5}$$

is the metric tensor with the metric signature $(+, -, -, -)$ (time-like or particle physics convention).

3. Apply small variations to the field $\varphi(x) \to \varphi(x) + \delta\varphi(x)$ to derive the Euler-Lagrange equation of motion for the field $\varphi(x)$. As in the classical paradigm, the deformations $\delta\varphi$ vanish on the spatial boundary of the 4-dimensional space-time region.

It is known from Special Relativity that physical processes do not depend on the reference system and thus Lagrangian densities have to be invariant under Lorentz transformations. It is possible to show that the action S formed by integrating $\mathcal{L}$ over space-time and equation of motions are Lorentz invariant.

For a discrete system, the Hamiltonian (total energy of the system) is defined as the spatial integral of the Hamiltonian density

$$H = \int dx \mathcal{H} = \sum p\dot{q} - L \tag{6}$$

for each dynamical variable q with conjugate momentum $p \equiv \partial L/\partial \dot{q}$, $\dot{q} = \partial L/\partial t$.

4. How can the Hamiltonian be generalized to a continuous system? Introduce the momentum density conjugate $\pi(\boldsymbol{x}) = \partial \mathcal{L}/\partial \dot{\varphi}(\boldsymbol{x})$ to $\varphi(\boldsymbol{x})$.

Scalar Fields

The most simple example is a theory of a single, scalar and real-valued field $\varphi(x)$ governed by the Lagrangian density

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 & (7) \\ &= \frac{1}{2}\left(\partial_\mu\phi\, \partial^\mu\phi - m^2\phi^2\right) & (8) \end{aligned}$$

where m is a constant.

1. Derive the equation of motion for this Lagrangian using the usual procedure.
2. Which kind of systems can this equation describe? How can parameter "m" be interpreted for such systems?
3. What is the solution for a free scalar particle?
4. Noting that the canonical momentum density conjugate to the field $\varphi(x)$ is $\dot{\varphi}(x) = \pi(x)$, what is the constructed Hamiltonian?

Conservation Laws

Noether's theorem summarizes the relationship between symmetries and conservation laws in classical field theory, stating that for each symmetry of a physical system under continuous transformations on the field ϕ there is a conserved quantity. Consider the transformation $\varphi(x) \to \varphi(x) + \alpha\Delta\varphi(x)$, α is an infinitesimal parameter and $\Delta\varphi$ is some deformation of the field.

1. Show that for each continuous symmetry of the Lagrangian density $\mathcal{L}$ there is a current $j^\mu(x)$ conserved, that is

$$j^\mu = \frac{\partial L}{\partial(\partial_\mu\phi)}\Delta\phi - J^\mu \quad \text{where} \quad \Delta L = \partial_\mu J^\mu \ . \tag{9}$$

If a transformation leaves the equation of motion invariant, i.e. $\Delta L \to 0$, then it can be called "symmetry".

2. Trivial example: consider the Lagrangian with only a kinetic form $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2$ under the transformation $\phi \to \phi + \alpha$. Show that the conserved current is

$$j^\mu = \partial^\mu\phi \ . \tag{10}$$

3. Less trivial example: consider the Lagrangian

$$\mathcal{L} = |\partial_\mu\varphi|^2 - m^2|\varphi|^2 \tag{11}$$

where ϕ is a complex-valued field. Show that the equation of motion is the Klein-Gordon equation. Show that this Lagrangian is invariant under the transformation $\phi \to e^{i\alpha}\phi$, $\alpha \in \mathbb{R}$, by treating the fields ϕ and ϕ^* as independent. Show that the conserved Noether current is now

$$j^\mu = i\left[(\partial^\mu\varphi^*)\varphi - \varphi^*(\partial^\mu\varphi)\right] \ . \tag{12}$$

This conserved current j^μ will be interpreted later as the electromagnetic current density carried by the scalar field ϕ which couples to an electromagnetic one, and the spatial integral of j^0 as its electric charge.

Dirac Equation

In the lectures you have seen how the Dirac equation and its solutions can be derived. The Dirac equation can be also obtained by applying the Euler-Lagrange equation. Consider the Dirac Lagrangian density

$$\mathcal{L} = \bar{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi, \quad \bar{\psi} \equiv \psi^{\dagger}\gamma^{0} \quad \text{(natural units } \hbar = c = 1\text{)}. \tag{13}$$

1. Insert this Lagrangian density into the Euler-Lagrange equations and derive the Dirac equation.
2. What does it describe and what kind of objects are the ψ functions?
3. Show that the Dirac equation implies the Klein-Gordon equation

$$\left(\Box + m^2\right)\psi(x) = 0\ . \tag{14}$$

The γ matrices can be used along with Dirac spinors to make a Lorentz scalar, pseudo-scalar, vector, axial vector and tensor. In general, one can build a complete set of *bilinear covariants*, which could be used together to make up Lagrangian functions to describe physical quantities.

In the next tutorial, we will see how a Dirac spinor transforms when we move from one inertial system to another, and how the different Dirac spinor products can be assembled in various linear combinations to construct quantities with distinct transformation behavior.

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