Tutorial 3

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"Tests des Standardmodells der Teilchenphysik"

by *PD Dr. Hubert Kroha*
Max-Planck-Institut für Physik, München

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Dr. Zinonas Zinonos

Maxwell's Equations - Covariant Formulation of Classical Electromagnetism

Let's begin by considering the basic laws of classical electromagnetism, the *Maxwell equations*. For simplicity, we can use a system of units (Heaviside–Lorentz) which is very convenient and practical in particle physics: $\mu_0 = \epsilon_0 = c = \hbar = 1$ [1]. Before Maxwell's work these laws were

$$\nabla \cdot \mathbf{E} = \rho_{\text{em}} \qquad \text{Gauss' law} \tag{1}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathrm{B}}{\partial t} \qquad \text{Faraday-Lenz laws} \tag{2}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{Absence of magnetic charges} \tag{3}$$

$$\nabla \times \mathbf{B} = \mathbf{j}_{\text{em}} \qquad \text{Ampère's circuital law (steady currents)}\ . \tag{4}$$

Here ρ_{em} is the charge density and $\mathbf{j}_{\text{em}}$ stands for the current density; these densities act as 'sources' for the $\mathbf{E}$ and $\mathbf{B}$ fields. Maxwell realized that taking the divergence of this last equation leads to conflict with the continuity equation for electric charge

$$\frac{\partial \rho_{\text{em}}}{\partial t} + \nabla \cdot \mathbf{j}_{\text{em}} = 0 \tag{5}$$

since [2]

$$\nabla \cdot (\nabla \times \mathbf{B}) = 0 \tag{6}$$

and hence from Eq. 4 it follows that

$$\nabla \cdot \mathbf{j}_{\text{em}} = 0 \tag{7}$$

This can only be true in situations where the charge density is constant in time. But for the general case, Maxwell modified Ampère's law to read

$$\nabla \times \mathbf{B} = \mathbf{j}_{\text{em}} + \frac{\partial \mathbf{E}}{\partial t} \tag{8}$$

which is now consistent with Eq. 5. The extra term introduced by Maxwell, the "electric displacement current" owes its place in the dynamical equations to a local conservation requirement. Therefore, Equations 1-3, together with 8, constitute Maxwell's equations in free space (apart from the sources).

1. Now, in a region with charges ($\rho_{\text{em}} \neq 0$) and currents ($\boldsymbol{J} \neq \mathbf{0}$), derive the four differential Maxwell equations from the following Lagrangian density:

$$\mathcal{L} = \mathcal{L}_{\text{field}} + \mathcal{L}_{\text{int}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu\,, \tag{9}$$

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[1] The relation between the universal constants ϵ_0, permittivity of free space, and μ_0, the permeability of free space, is given by $\epsilon_0 \mu_0 = 1/c^2$.

[2] The vitally important continuity Eq. 5 states that the rate of decrease of charge e in any arbitrary volume V is due precisely and only to the flux of current out of its surface. That is, no net charge can be created or destroyed in V. Since V can be made as small as we desire, this means that electric charge e must be locally conserved. So, a process in which charge e is created at one point and destroyed at a distant one is not allowed, despite the fact that it conserves the charge overall or 'globally'. The ultimate reason for this is that the global form of charge conservation would necessitate the instantaneous propagation of signals (such as "create a positron over there"), and this conflicts with special relativity.

where $A^\mu = \left(A^0, \boldsymbol{A}\right)$ with $A^0 = \Phi$ is the electromagnetic vector potential, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor,

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ +E_1 & 0 & -B_3 & +B_2 \\ +E_2 & +B_3 & 0 & -B_1 \\ +E_3 & -B_2 & +B_1 & 0 \end{pmatrix} , \tag{10}$$

while $j^\mu = \left(j^0, \boldsymbol{J}\right)$ with $j^0 = \rho_{\text{em}}$ and $\boldsymbol{J} = \mathbf{j}_{\text{em}}$ constitute the four-vector current density. It is convenient to introduce the vector potential $A_\mu(x)$ in place of the fields $\mathbf{E}$ and $\mathbf{B}$

$$\mathbf{A} = \nabla \times \mathbf{B} \tag{11}$$

and

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} . \tag{12}$$

2. How do the electric $\boldsymbol{E}$ and magnetic $\boldsymbol{B}$ fields transform under Lorentz boosts?
3. How do the electric $\boldsymbol{E}$ and magnetic $\boldsymbol{B}$ fields transform under space rotations?
4. How do the electric $\boldsymbol{E}$ and magnetic $\boldsymbol{B}$ fields mix in each case?

Show these transformations starting from the $F^{\mu\nu}$ field tensor in the matrix form.

5. Try to add a mass term for the photon field in Equation 9 and explain why such a term is not allowed for this field.

References

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