Tutorial 4

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"Tests des Standardmodells der Teilchenphysik"

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Symmetry Groups

In elementary particle physics, the most common groups are of the type $U(n)$; the collection of all unitary $n \times n$ matrices[1]. If we restrict ourselves further to unitary matrices with determinant 1, then the group is called $SU(n)$.

1. Show that the set of all unitary $n \times n$ matrices constitutes a group. To prove *closure*, for instance, it must be shown that the product of two unitary matrices is itself unitary.

2. Show that the set of all unitary $n \times n$ matrices with determinant 1 constitutes a group.

If we limit ourselves to *real* unitary matrices, the group is $O(n)$[2]. Finally, the group of real, orthogonal, $n \times n$ matrices with determinant 1 is $SO(n)$, which may be thought of as the group of all *rotations* in a space of n dimensions ($SO(3)$ describes the rotational symmetry related to the Noether's theorem to the conservation of angular momentum).

3. Show that $O(n)$ is a group.

4. Show that $SO(n)$ is a group.

Rotation Groups In Action

In general, if the coordinate axes of a system are rotated the components of a vector within that system will change. Let's consider a vector $\mathbf{A}$ in two dimensions and suppose its components with respect to Cartesian axes $\{x,\, y\}$ are $(\alpha_x,\, \alpha_y)$.

1. What are its transformed components $(\alpha'_x,\, \alpha'_y)$ in a system $\{x',\, y'\}$, which is rotated counterclockwise by an angle θ? The answer can be expressed in the form if a 2×2 matrix $R(\theta)$:

$$\begin{pmatrix} \alpha'_x \\ \alpha'_y \end{pmatrix} = R \begin{pmatrix} \alpha_x \\ \alpha_y \end{pmatrix} . \tag{1}$$

2. Show that the R representation is an *orthogonal* matrix and then find its determinant.

3. The set of all such rotations described by matrix R constitutes a group; how is it called?

4. By multiplying the matrices, show that $R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$; is this an Abelian group?

Consider the matrix

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{2}$$

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[1] A unitary matrix is one whose inverse is equal to its transpose conjugate, $U^{-1} = \tilde{U}^*$

[2] An orthogonal matrix is one whose inverse is equal to its transpose, $O^{-1} = \tilde{O}$

5. Does it belong to the $O(2)$ group? How about $SO(2)$? What is the effect on vector $\boldsymbol{A}$? Does it describe a possible rotation of the $x-y$ plane?

Now, we might inquire how the components of a *spinor*

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \alpha\begin{pmatrix}1\\ 0\end{pmatrix} + \beta\begin{pmatrix}0\\ 1\end{pmatrix} \tag{3}$$

transforms under rotations. The transformation follows the following rule

$$\begin{pmatrix}\alpha'\\ \beta'\end{pmatrix} = U(\boldsymbol{\theta})\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \tag{4}$$

where $U(\boldsymbol{\theta})$ is the 2×2 matrix

$$U(\boldsymbol{\theta}) = e^{-\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}\,. \tag{5}$$

In this representation, the vector $\boldsymbol{\theta}$ points along the axis of rotation and its magnitude θ is the angle of rotation about that axis (right-hand rule) and $\hat{\theta} = \boldsymbol{\theta}/\theta$. The "Pauli spin matrices" $\boldsymbol{\sigma}$ arise in Wolfgang Pauli's treatment of spin in quantum mechanics and comprise a set of three 2×2 matrices which are Hermitian and unitary. They are defined by

$$\begin{aligned}\sigma_1 = \sigma_x &= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\\ \sigma_2 = \sigma_y &= \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\\ \sigma_3 = \sigma_z &= \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\end{aligned} \tag{6}$$

so that $\mathbf{S} = \frac{\hbar}{2}\boldsymbol{\sigma}$. Therefore, for a spin-1/2 particle, the spin operator is given by $\boldsymbol{S}$ the fundamental representation of the $SU(2)$ group. The Pauli matrices are occasionally denoted by $\boldsymbol{\tau}$ when used in connection with *isospin* symmetries.

We realize that the exponent in (5) is itself a matrix, so an expression of this form can be interpreted as a shorthand for the power series

$$e^A \equiv 1 + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots \tag{7}$$

6. Show that $e^{i\pi\sigma_z/2} = i\sigma_z$.

7. Find the matrix $\mathcal{U}$ representing a rotation by 180° about the y axis and show that it is able to convert a "spin up" state into a "spin down" one, as we would expect.

8. More generally, we can show that

$$\mathcal{U}(\boldsymbol{\theta}) = \cos\frac{\theta}{2} - i(\hat{\theta}\cdot\boldsymbol{\sigma})\sin\frac{\theta}{2} \tag{8}$$

   Use the identity

$$(\boldsymbol{\sigma}\cdot\boldsymbol{a})(\boldsymbol{\sigma}\cdot\boldsymbol{b}) = (\boldsymbol{a}.\boldsymbol{b})I + i\boldsymbol{\sigma}.(\boldsymbol{\alpha}\times\boldsymbol{b})\,, \tag{9}$$

   which holds for any $\boldsymbol{a}$ and $\boldsymbol{b}$ "good" vectors. Prove it!

9. Show that $\mathcal{U}(\boldsymbol{\theta})$ is a unitary matrix of determinant 1. In fact, the full set of such rotation matrices makes up the $SU(2)$ transformation group.

Therefore, spin-1/2 particles (leptons, quarks, baryon octet N, Λ, $\Sigma^{\pm,0}$, $\Xi^{-,0}$, Λ_c^+) transform under rotations according to the two-dimensional representation of $SU(2)$. Likewise, spin-1 particles (mediators, vector mesons ρ, K^*, ω, ψ, D^*, Υ), described by *vectors*, belong three-dimensional representation of $SU(2)$; spin-3/2 particles (baryon decuplet Δ, Σ^*, Ξ^*, Ω^-) described by a four-component object transform under the four-dimensional representation of $SU(2)$ and so on. Particles of different spin, belong to different *representations* of the rotation group.

One may wonder how is $SU(2)$ (the most important internal symmetry in elementary particle physics) related to rotations; $SU(2)$ is essentially very similar to the mathematical structure of $SO(3)$, the group of rotations in three dimensions. However, there's a subtle difference between $SU(2)$ and $SO(3)$. According to the previous problem, the matrix U for rotation through an angle 2π is -1, that is a spinor *changes sign* under such rotation. On the other hand, geometrically, a rotation of a system through 2π is equivalent to "no rotation" at all. $SU(2)$ is thus a kind of "doubled" version of the $SO(3)$ group in which spin-1/2 particles must be rotated by an angle of 4π in order to return to their original configuration. In this sense, spinor representations of $SU(2)$ are not "true" representations of the rotation group and hence they do not appear in classical mechanics. In quantum mechanics, only the square of the wave function carries physical significance, and being such the minus sign goes away.

Finally, the Pauli matrices, after a multiplication by i to make them anti-Hermitian, also generate transformations in the sense of *Lie algebras*. The matrices $i\sigma_j$ with $j = 1, 2, 3$ form a basis for $SU2$), which exponentiates to the special unitary group $SU(2)$:

$$SU(2) = \text{span}\left\{\frac{i\sigma_1}{2}, \frac{i\sigma_2}{2}, \frac{i\sigma_3}{2}\right\}. \tag{10}$$

As a result, each $i\sigma_i$ can be seen as an infinitesimal generator of $SU(2)$. The elements of $SU(2)$ are then exponentials of linear combinations of these three generators, and multiply as indicated above in (9) and as follows:

$$\sigma_i \sigma_j = i\varepsilon_{ijk}\, \sigma_k + \delta_{ij} I \ . \tag{11}$$

Addendum

The set of all symmetry operations on a particular system has the following properties:

- Closure: If R_i and R_j are in the set, then the ordered product $R_i R_j$ is also in the same set. That is, there exists some R_k such that $R_i R_j = R_k$.
- Identity: There is an element I such that $IR_i = R_i I = R_i$ for all elements R_i.
- Inverse: For every element R_i there is an *inverse*, R_i^{-1} such that $R_i R_i^{-1} = R_i^{-1} R_i = I$.
- Associativity: $R_i(R_j R_k) = (R_i R_j)R_k$.

The above are the defining properties of a mathematical *group*. If all the elements of the group commute, $R_i R_j = R_j R_i$, then the group is called *Abelian*.