Tutorial 6

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"Tests des Standardmodells der Teilchenphysik"

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Spontaneous Breaking of a Local $SU(2)$ Gauge Symmetry

The spontaneous breaking of a $U(1)$ gauge symmetry has been demonstrated in the last tutorial. The same procedure is repeated here but for an $SU(2)$ gauge symmetry, encapsulating all the principles we have introduced, as well as providing preparation for the discussion of the gauge symmetries of electroweak interactions.

The non-Abelian [1] group $SU(2)$ is the set of two dimensional, complex unitary matrices with unit determinant. The unit determinant constraint removes one more parameter and this group has then three generators and three parameters.

1. Write down the generators of the $SU(2)$ group, as a set of three linearly independent, traceless 2×2 Hermitian matrices, consisting the elements of the *Lie algebra*.

Consider now the Lagrangian

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2 \tag{1}$$

which includes an $SU(2)$ *doublet* of complex scalar fields

$$\phi(x) = \begin{pmatrix} \phi_\alpha \\ \phi_\beta \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} \,, \tag{2}$$

where ϕ_i are real-valued fields. The potential term is apparently

$$V(\phi) = \mu^2 \phi^\dagger \phi + \lambda \left(\phi^\dagger \phi\right)^2 \tag{3}$$

2. Find the minimum of the potential in the case $\mu^2 < 0$ and $\lambda > 0$.

The Lagrangian is manifestly invariant under $SU(2)$ global gauge transformations, namely

$$\phi \to \phi' = e^{i\alpha_i \tau_i/2} \quad \text{with} \quad i = 1,\, 2,\, 3\,. \tag{4}$$

To achieve local invariance under $SU(2)$ phase transformations, that is $\alpha_i = \alpha_i(x)$, we need to replace the partial derivative ∂_μ by the covariant derivative

$$D_\mu = \partial_\mu + ig\frac{\tau_\alpha}{2} W_\mu^\alpha \,, \tag{5}$$

and consider the three gauge fields $W_\mu^i(x)$ with $i = 1,\, 2,\, 3$ ($SU(2)$ vectors), which under the infinitesimal gauge transformation

$$\phi(x) \to \phi'(x) = (1 + i\boldsymbol{\alpha}(x) \cdot \frac{\boldsymbol{\tau}}{2})\phi(x) \tag{6}$$

transform as follows

$$\boldsymbol{W}_\mu \to \boldsymbol{W}'_\mu = \boldsymbol{W}_\mu - \frac{1}{g}\partial_\mu \boldsymbol{\alpha} - \boldsymbol{\alpha} \times \boldsymbol{W}_\mu \,. \tag{7}$$

[1] If the generators do not commute with one another, the group is called "non-Abelian".

Basically, the $\boldsymbol{\alpha} \times \boldsymbol{W}_\mu$ term arises because $\boldsymbol{W}_\mu$ is an $SU(2)$ gauge vector; it is being *rotated* even if phase $\boldsymbol{\alpha}$ can be independent of x.

The *locally* gauge invariant Lagrangian is then

$$\mathcal{L} = \left(\partial_\mu \phi + i\frac{g}{2}\boldsymbol{\tau} \cdot \boldsymbol{W}_\mu \phi\right)^\dagger \left(\partial^\mu \phi + i\frac{g}{2}\boldsymbol{\tau} \cdot \boldsymbol{W}^\mu \phi\right) - V(\phi) - \frac{1}{4}\boldsymbol{W}_{\mu\nu} \cdot \boldsymbol{W}^{\mu\nu} \ , \tag{8}$$

where the last term with

$$\boldsymbol{W}_{\mu\nu} = \partial_\mu \boldsymbol{W}_\nu - \partial_\nu \boldsymbol{W}_\mu - g\, \boldsymbol{W}_\mu \times \boldsymbol{W}_\nu \tag{9}$$

represents the kinetic energy of the gauge vector fields. It is a consequence of the non-Abelian character of the $SU(2)$ group; that is, it occurs because the $\boldsymbol{\tau}$ matrices do not commute with each other.

If $\mu^2 > 0$, the Lagrangian (8) describes a system of four scalar particles as shown in (2), each of mass μ, interacting with three massless gauge bosons, $\boldsymbol{W}_\mu^\alpha$.

However, we are interested in the case $\mu^2 < 0$ and $\lambda > 0$. The potential $V(\phi)$ is then getting a minimum at a finite value of $|\phi|$ where

$$\phi^\dagger \phi = \frac{1}{2}\left(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2\right) = -\frac{\mu^2}{2\lambda} \, . \tag{10}$$

In order to expand $\phi(x)$ about a particular energy minimum, we can choose the set

$$\phi_1 = 0, \quad \phi_2 = 0, \quad \phi_3^2 = \upsilon^2, \quad \phi_4 = 0 \tag{11}$$

whose effect is equivalent to the spontaneous breaking of the $SU(2)$ symmetry, where the symmetry is merely "hidden" by our choice of ground state. The expansion of $\phi(x)$ can be performed about this particular ground state (the vacuum)

$$\phi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \upsilon \end{pmatrix} \ . \tag{12}$$

Due to the gauge invariance, we can simply substitute the expansion

$$\phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \upsilon + h(x) \end{pmatrix} \ , \tag{13}$$

where $h(x)$ describes a massive scalar spin-0 field, the Higgs boson. As in the $U(1)$ spontaneous symmetry breaking, we can parametrize the fluctuations from the vacuum ϕ_0 in terms of four real fields θ_i, $i = 1, 2, 3$ and $h(x)$ as follows

$$\phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \upsilon + h(x) \end{pmatrix} e^{i\boldsymbol{\tau}\cdot\boldsymbol{\theta}(x)/\upsilon} \tag{14}$$

3. Show that for small perturbations, the four fields are independent and thus fully parametrize the deviations from the vacuum ϕ_0.

Consequently, by gauging the three massless Goldstone boson fields $\boldsymbol{\theta}(x)$, the Lagrangian does contain no trace of the $\boldsymbol{\theta}(x)$ fields and is locally $SU(2)$-transformation invariant; only the Higgs field $h(x)$ remains out of the four fields.

Now, we want to determine the masses generated for the gauge bosons W_μ^α by substituting ϕ_0 of Equation 12 in the Lagrangian of Equation 8.

4. Identify the relevant term for the boson masses in the Lagrangian of Equation 8 and insert the ground state of the field $\phi(x)$. Calculate the gauge boson masses as a function of the minimum potential υ. Show that their mass is given by

$$M = \frac{1}{2} g \upsilon \ . \tag{15}$$

The Lagrangian hence describes *three* massive gauge bosonic fields $\boldsymbol{W}$ and *one* massive scalar field h. So, we can say that the gauge fields have "eaten up" the massless Goldstone bosons and become massive. The scalar degrees of freedom (Goldstone bosons) transform into longitudinal polarizations of the massive vector bosons; this is another example of the Higgs mechanism.

5. Determine the value of υ. Use the Fermi's constant $G_F = 1.16637 \times 10^{-5}$ GeV^{-2}.

Masses of Gauge Bosons - Choice of the Higgs Field in the Glashow-Weinberg-Salam Model

To generate the masses for the gauge bosons in a gauge invariant way within the Glashow-Weinberg-Salam Model, we must use the Higgs mechanism extended to the $SU(2)\times U(1)$ symmetry. The Higgs mechanism must be formulated in a way such that the bosons $W^{\pm}$ and Z^0 will acquire mass and the photon A_μ will remain massless. To achieve this, Weinberg suggested in 1967 the $SU(2)\times U(1)$ gauge invariant Lagrangian

$$\mathcal{L} = \left| \left(i\partial_\mu - g\,\boldsymbol{T}\cdot\boldsymbol{W_\mu} - g'\,\frac{Y}{2}\,B_\mu \right)\phi \right|^2 - V(\phi) \tag{16}$$

where $|\cdots|^2 \equiv (\cdots)^\dagger(\cdots)$ and with the potential term being

$$V(\phi) = \mu^2\phi^\dagger\phi + \lambda\left(\phi^\dagger\phi\right)^2 \;. \tag{17}$$

The operators $\boldsymbol{T} = \boldsymbol{\tau}/2$ and Y are the generators of the $SU(2)_L$ and $U(1)_Y$ groups of gauge transformations, respectively, so that the $SU(2)\times U(1)$ transformations on the *left-hand* (LH) and *right-hand* (RH) components of the fermionic ψ wavefunction are

$$\chi_L \to \chi'_L \;=\; e^{i\boldsymbol{\alpha}(x).\boldsymbol{T}+i\beta(x)Y}\;\chi_L \tag{18}$$

$$\psi_R \to \psi'_R \;=\; e^{i\beta(x)Y}\;\psi_R\,. \tag{19}$$

The weak hypercharge [2], satisfies the equality:

$$Q = T_3 + \frac{Y}{2}\;, \tag{20}$$

where Q is the electrical charge and T_3 stands for the third component of weak isospin $\mathbf{T}$ [3], so that

$$j_\mu^{EM} = J_\mu^3 + \tfrac{1}{2}j_\mu^Y\;. \tag{21}$$

In other words, the three vector bosons $\boldsymbol{W}_\mu$ are coupled to the isotriplet of weak currents $\boldsymbol{J}_\mu$

$$-\,ig\;\boldsymbol{J}_\mu\cdot\boldsymbol{W}^\mu = -ig\;\bar\chi_L\;\gamma_\mu\;\boldsymbol{T}\cdot\boldsymbol{W}^\mu\;\chi_L \qquad : SU(2)_L \tag{22}$$

and the vector boson B_μ couples to the weak hypercharge currents

$$-\,i\frac{g'}{2}j_\mu^Y B^\mu = -ig'\bar\psi\gamma_\mu\frac{Y}{2}\psi B^\mu \qquad\qquad : U(1)_L\;. \tag{23}$$

The LH fermions form isospin doublets

$$\chi_L \;=\; \begin{pmatrix}\nu_\ell\\ \ell^-\end{pmatrix}_L \quad \text{for leptons with } T=\tfrac{1}{2} \text{ and } Y=-1 \tag{24}$$

$$\chi_L \;=\; \begin{pmatrix}u\\ d\end{pmatrix}_L \quad \text{for quarks with } T=\tfrac{1}{2} \text{ and } Y=\tfrac{1}{3} \tag{25}$$

and the RH fermions are isosinglets

$$\psi_R \;=\; \ell_R^- \text{ for leptons with } T=0 \text{ and } Y=-2 \tag{26}$$

$$\psi_R \;=\; u_R,\, d_R \text{ for quarks with } T=0 \text{ and } Y=\tfrac{4}{3},\,-\tfrac{2}{3}\,. \tag{27}$$

In order to keep $\mathcal{L}$ invariant, Weinberg introduced an isospin doublet with weak hypercharge $Y=1$ of complex scalar fields,

$$\phi = \begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_1+i\phi_2\\ \phi_3+i\phi_4\end{pmatrix}\;. \tag{28}$$

To generate gauge boson masses, we use the familiar Higgs potential $V(\phi)$ of Equation 17 with $\mu^2<0$ and $\lambda>0$, and choose a vacuum expectation value ϕ_0 of $\phi(x)$

$$\phi_0 \equiv \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ \upsilon\end{pmatrix}\;. \tag{29}$$

[2] The weak hypercharge Y is the generator of the $U(1)$ component of the electroweak gauge group, $SU(2)\times U(1)$, and its associated quantum field B mixes with the $W^{3\,\mu}$ electroweak quantum field to produce the observed Z gauge boson and the photon of quantum electrodynamics.

[3] The weak isospin conservation law relates the conservation of the "3rd component of weak isospin" $\mathbf{T}$, T_3. All weak interactions must preserve T_3 and is therefore a conserved quantity.

But why is an isospin doublet of complex scalar fields, with $Y = 1$ and vacuum expectation value of ϕ_0 of (29), suitable for the problem in hand? Any choice of ϕ_0 which breaks a symmetry operation would inevitably generate a mass for the corresponding gauge boson. However, if the vacuum ϕ_0 is still left invariant by some subgroup of gauge transformations, then the gauge bosons associated with this subgroup will unfortunately remain massless. In our case, the choice ϕ_0 with $T = \frac{1}{2}$, $T_3 = -\frac{1}{2}$ and $Y = 1$ breaks both $SU(2)$ and $U(1)_Y$ gauge symmetries. But since ϕ_0 is neutral, the $U(1)_{\rm em}$ symmetry with generator $Q = T_3 + \frac{Y}{2}$ remains unbroken. That is,

$$Q\ \phi_0 = 0\,, \tag{30}$$

so that

$$\phi_0 \to \phi_0' = e^{i\alpha(x)Q}\phi_0 \simeq (1 + i\alpha(x)Q)\phi_0 = \phi_0 \tag{31}$$

for any value of $\alpha(x)$.The vacuum is thus invariant under $U(1)_{\rm em}$ transformations, and the photon remains massless. Out of the four $SU(2)\times U(1)_Y$ generators $\boldsymbol{T}$ and Y, only the combination Q obeys relation (30). The other three generators break the symmetry and generate massive gauge bosons, as we wanted.

Now, we wish to determine the gauge boson masses by substituting the vacuum expectation ϕ_0 for the field $\phi(x)$ in the Lagrangian.

1. Determine the mass of the Higgs boson in terms of the parameters μ, υ, λ.
2. Show that the relevant mass term can take the form

$$\left(\frac{1}{2}\upsilon g\right)^2 W_\mu^+ W^{-\mu} + \frac{1}{8}\upsilon^2 \begin{pmatrix} W_\mu^3 & B_\mu \end{pmatrix} \begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix} \begin{pmatrix} W^{3\mu} \\ B^\mu \end{pmatrix} \tag{32}$$

3. Identify the mass for the charged bosons

$$W_\mu^\pm = \frac{W_\mu^1 \mp i W_\mu^2}{\sqrt{2}}\ . \tag{33}$$

4. Find the eigenvalues and eigenstates of the 2×2 matrix in (33) corresponding to the neutral vector bosons.
5. Re-express the previous results in terms of the Weinberg/weak angle θ_W using the relation

$$\frac{g'}{g} = \tan\theta_W\ . \tag{34}$$

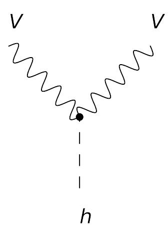
6. Show that the masses of the massive gauge bosons are related though the equation

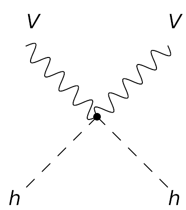
$$\frac{M_W}{M_Z} = \cos\theta_W \tag{35}$$

7. Explain why the inequality $M_W \neq M_Z$ occurs and thus the Weinberg-Salam model with a Higgs doublet fixes the parameter ρ, which specifies the relative strength of the *neutral* and *charged* current weak interactions, is set to unity,

$$\rho \equiv \left(\frac{M_W}{M_Z\ \cos\theta_W}\right)^2 = 1\ . \tag{36}$$

8. The Lagrangian for the scalar field (16) contains trilinear $h\ W^+W^-$ and quadrilinear $h\ h\ W^+W^-$ Higgs boson couplings, namely





Using

$$\phi_0 \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \upsilon + h(x) \end{pmatrix} \tag{37}$$

show that these vertex factors in the Standard Model are igM_W and $i\frac{1}{4}g^2$, respectively.

9. Determine the hZZ and $hhZZ$ vertex factors.
10. What is the coupling of the Higgs boson to the photons?

Masses of the Fermions

The masses for the gauge vector bosons, as formulated in the Weinberg-Salam model, have been derived in the previous exercise. Now we follow a similar approach to generate the mass for the leptons. The Lagrangian describing the lepton kinetic energies and their interaction with $W^{\pm}$, Z^0 and γ is

$$\mathcal{L} = \bar{L}\gamma^{\mu}\left(i\partial_{\mu} - g\frac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{W}_{\mu} - g'\frac{Y}{2}B_{\mu}\right)L\ , \tag{38}$$

in an $SU(2)\times U(1)_Y$ invariant form. The L denotes a left-handed lepton doublet (for example, $L = \begin{pmatrix}\nu_\ell \\ \ell^-\end{pmatrix}_L$ with $T = 1/2$ and $Y = -1$) and R stands for a right-handed fermion singlet (for example, $R = \ell_R^-$).

1. Why mass terms such as $-m_\ell\bar{\psi}\psi$ cannot be added in the Lagrangian?

2. In order to generate mass for the leptons, the following $SU(2)\times U(1)$ gauge invariant term can be included in the Lagrangian
$$\mathcal{L}' = -G_\ell\left[\bar{L}\phi R + \bar{R}\phi^{\dagger}L\right]\ , \tag{39}$$
where G_ℓ is an ad-hoc, arbitrary coupling constant. Apply the idea of spontaneously breaking the symmetry by embedding
$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix}0 \\ v + h(x)\end{pmatrix} \tag{40}$$
into the Lagrangian (39), leading thus to the generation of the required lepton masses m_ℓ. Identify the mass of the leptons.

3. Compare the couplings of the leptons and the charged gauge vector bosons to the Higgs boson, and explain why the former are so small.

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