Tutorial 7

January 11, 2017

"Tests des Standardmodells der Teilchenphysik"

by *PD Dr. Hubert Kroha*
Max-Planck-Institut für Physik, München

WS 2016

Dr. Zinonas Zinonos

The electron-positron scattering process, $e^+e^- \to \mu^+\mu^-$ - Part 1/2

The process $e^+e^- \to \mu^+\mu^-$ is considered as one of the simplest and fundamental of all QED processes, but also one of the most important in high-energy physics. We will compute the *unpolarized* cross-section for $e^+e^- \to \mu^+\mu^-$ to lowest order.

The general procedure to calculate such unpolarized cross-sections for QED processes is described below.

1. First of all, we draw all possible Feynman diagrams for the process in question. For this process, the corresponding diagram is shown in Figure 2. We apply the Feynman rules to write down an expression of

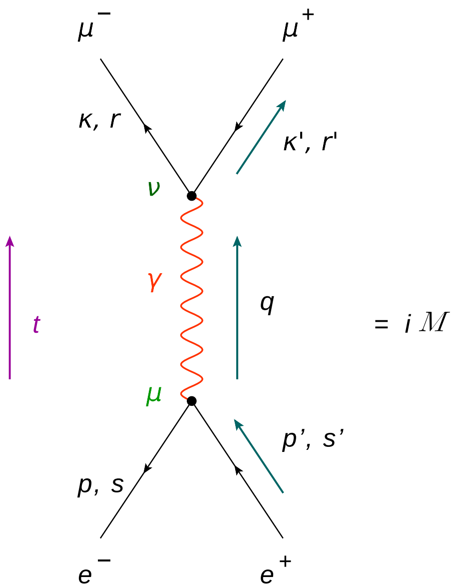


Figure 1: Feynman diagram for the QED process $e^+e^- \to \mu^+\mu^-$.

   the amplitude $\mathcal{M}$ (matrix element).

2. The next step is to square the matrix element $i\mathcal{M}$. Before squaring the matrix element, show the following useful relations, for the bispinor products that are involved in the $|\mathcal{M}|^2 = (i\mathcal{M})(i\mathcal{M})^*$ expression, can hold

$$(\bar{u}\gamma_\mu \upsilon)^* = \bar{\upsilon}\gamma_\mu u \quad (1)$$
$$(\bar{\upsilon}\gamma_\mu u)^* = \bar{u}\gamma_\mu \upsilon \; . \quad (2)$$

   Note that the γ^μ matrices obey the anti-commutation relations

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \; . \quad (3)$$

The γ^0 matrix is hermitian while the $\gamma^1, \gamma^2, \gamma^3$ matrices are anti-hermitian, $(\gamma^0)^\dagger = +\gamma^0$ while $(\gamma^i)^\dagger = -\gamma^i$ for $i = 1, 2, 3$. Find an expression for the squared matrix element $|\mathcal{M}|^2$.

3. To proceed further with the calculation of the unpolarized cross-section, we will have to:

   - *average* over the spin states s, s' of the incoming fermions assuming unpolarized initial states ($s_{e^\pm}$ have possible values $\pm\frac{1}{2}$)
   - *sum* over the spins states r, r' of the outgoing fermions (this case the muon states) since muon detectors are normally blind to polarization.

   In other words, we have to evaluate the following expression in order to obtain the spin-averaged amplitude of the process,

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{2s_{e^-}+1}\sum_{s=1}^{2}\frac{1}{2s_{e^+}+1}\sum_{s'=1}^{2}\sum_{r=1}^{2}\sum_{r'=1}^{2}|\mathcal{M}\,(s,\, s' \to r,\, r')\,|^2 \tag{4}$$

   To average/sum over spins we must use the completeness relations for the spinors u and v that can be found in the Addendum.

4. We then need to evaluate the traces of the Dirac matrices products using the trace properties, in order to simplify the expression of $|\mathcal{M}|^2$ [1]. Some of the most important properties of the γ matrices and the trace technology are listed in the Addendum. Show that the spin-averaged square matrix element will take the form

$$\langle|\mathcal{M}|^2\rangle = \frac{8e^4}{q^4}\left[(p'\cdot\kappa)(p\cdot\kappa') + (p'\cdot\kappa')(p\cdot\kappa) + m_e^2(\kappa\cdot\kappa') + m_\mu^2(p\cdot p') + 2m_e^2 m_\mu^2\right] \tag{5}$$

5. We now specialize a particular frame of reference and draw a picture of the kinematic variables in the center-of-mass frame in order to obtain a more explicit formula. The involved four-momentum vectors can

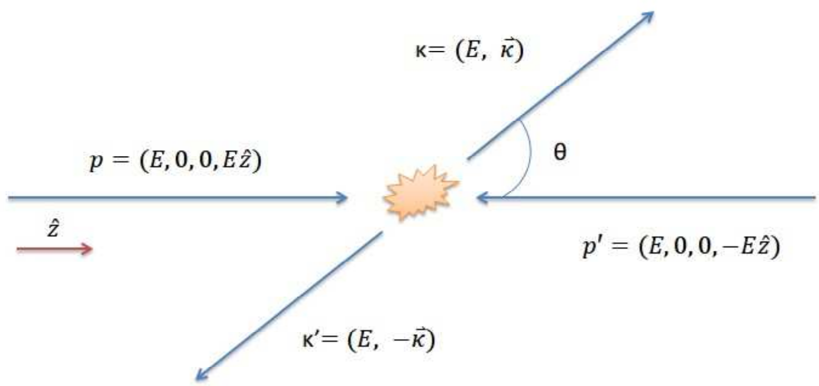


Figure 2: Kinematics for the QED process $e^+e^- \to \mu^+\mu^-$ in the center-of-mass frame.

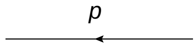
   be expressed in terms of the basic kinematic variables, energies and angles in that frame, as illustrated in Figure **??**. The expression of the spin-averaged scattering amplitude can be simplified by taking the mass of the electron to zero (much lighter with respect to the muon mass). Show that, when neglecting the electron masses, the spin-averaged square matrix element will take the form

$$\langle|\mathcal{M}|^2\rangle = e^4\left[\left(1+\left(\frac{m_\mu}{E}\right)^2\right) + \left(1-\left(\frac{m_\mu}{E}\right)^2\right)\cos^2\theta\right] . \tag{6}$$

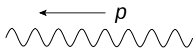
[1] Casimir's trick; a shortcut for finding a matrix element. Read 7.7 in Griffiths, D. J. *Introduction to Elementary Particles*. New York: Wiley, p. 251, 2013.

Addendum

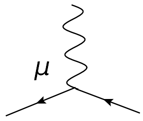
Feynman Rules for Quantum Electrodynamics

Dirac propagator: 

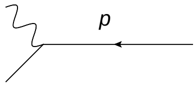
$$= \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} \quad (7)$$

Photon propagator: 

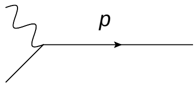
$$= \frac{-ig_{\mu\nu}}{p^2 + i\epsilon} \quad (8)$$

QED vertex: 

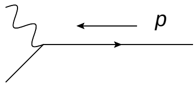
$$= iQe\gamma^{\mu} \quad (Q = -1 \text{ for an electron}) \quad (9)$$

External fermions: 

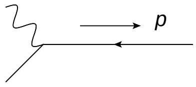
$$= u^s(p) \quad \text{(initial)} \quad (10)$$



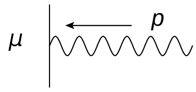
$$= \bar{u}^s(p) \quad \text{(final)} \quad (11)$$

External anti-fermions: 

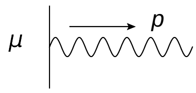
$$= \bar{v}^s(p) \quad \text{(initial)} \quad (12)$$



$$= v^s(p) \quad \text{(final)} \quad (13)$$

External photons: 

$$= \epsilon_{\mu}(p) \quad \text{(initial)} \quad (14)$$



$$= \epsilon^*_{\mu}(p) \quad \text{(final)} \quad (15)$$

Polarization rules of external fermions

The spinors $u^s(p)$ and $v^s(p)$ obey the Dirac equation in the form

$$(\not{p} - m)u^s(p) = \bar{u}^s(p)(\not{p} - m) = 0 \quad (16)$$
$$(\not{p} + m)v^s(p) = \bar{v}^s(p)(\not{p} + m) = 0 \quad (17)$$

where $\not{p} \equiv \gamma^{\mu}p_{\mu}$. The Dirac matrices obey the anti-commutation relations

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu} \; . \quad (18)$$

The “chiral” (or Weyl) basis is often used,

$$\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \bar{\sigma}^{\mu} & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad (19)$$

where

$$\sigma^{\mu} = (1, \boldsymbol{\sigma}), \quad \bar{\sigma} = (1, -\boldsymbol{\sigma}) \; . \quad (20)$$

In this basis, the normalized Dirac spinors can be written as follows

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\;\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\;\xi^s \end{pmatrix}, \quad v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\;\eta^s \\ -\sqrt{p\cdot\sigma}\;\eta^s \end{pmatrix} \quad (21)$$

where ξ^s and η^s are two-component spinors normalized to unity. In the ultra-relativistic limit, these expressions simplify to

$$u^s(p) \approx \sqrt{2E}\begin{pmatrix} \frac{1}{2}(1 - \hat{p}\cdot\boldsymbol{\sigma})\xi^s \\ \frac{1}{2}(1 + \hat{p}\cdot\boldsymbol{\sigma})\xi^s \end{pmatrix}, \quad v^s(p) \approx \sqrt{2E}\begin{pmatrix} \frac{1}{2}(1 - \hat{p}\cdot\boldsymbol{\sigma})\eta^s \\ -\frac{1}{2}(1 + \hat{p}\cdot\boldsymbol{\sigma})\eta^s \end{pmatrix} \quad (22)$$

Adopting the standard basis for the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{23}$$

we obtain, for example,

$$\xi^s = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{24}$$

for spin "up" in the z direction, and

$$\xi^s = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{25}$$

for spin "down" in the z direction. For anti-fermions, the physical spin is opposite to that of the spinor,

$$\eta^s = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{26}$$

corresponds to spin "down" in the z direction, and

$$\eta^s = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{27}$$

for spin "up" in the z direction.

In computing unpolarized cross-sections, one encounters the polarization sums

$$\sum_s u^s_\alpha(p)\bar{u}^s_\beta(p) = [(\not{p} + m)\mathbb{1}_4]_{\alpha\beta} \quad \text{and} \quad \sum_s v^s_\alpha(p)\bar{v}^s_\beta(p) = [(\not{p} - m)\mathbb{1}_4]_{\alpha\beta} \tag{28}$$

where

$$\not{p} = \gamma^\mu p_\mu \quad \text{and} \quad \bar{u} = u^\dagger\gamma^0\,. \tag{29}$$

and α, β are free Dirac indices.

For polarized cross-sections one can either resort to the explicit formulæ (21), or insert the projection matrices

$$\left(\frac{1+\gamma^5}{2}\right), \quad \left(\frac{1-\gamma^5}{2}\right) \tag{30}$$

which project onto *right*- and *left*-handed spinors, respectively. Again, for anti-fermions, the helicity of the spinor is opposite to the physical helicity of the particle.

Mathematical structure of the γ matrices

The defining property for the gamma matrices to generate a Clifford algebra is the anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}I_4 \tag{31}$$

where $\{,\}$ is the anticommutator, $\eta^{\mu\nu}$ is the Minkowski metric with signature $(+---)$ and I_4 is the 4×4 identity matrix. This defining property is more fundamental than the numerical values used in the specific representation of the gamma matrices. Covariant gamma matrices are defined by

$$\gamma_\mu = g_{\mu\nu}\gamma^\nu = \left\{\gamma^0, -\gamma^1, -\gamma^2, -\gamma^3\right\}, \tag{32}$$

and Einstein notation is assumed.

Numerator algebra

Trace identities Three of the main properties of the trace operator are:

$$\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B) \tag{33}$$
$$\mathrm{tr}(rA) = r\,\mathrm{tr}(A) \tag{34}$$
$$\mathrm{tr}(ABC) = \mathrm{tr}(CAB) = \mathrm{tr}(BCA) \tag{35}$$

Trace properties involving $\boldsymbol{\gamma}$ matrices Traces of γ matrices can be evaluated as follows:

$$\begin{aligned} \mathrm{tr}(\mathbf{1}) &= 4 && (36) \\ \mathrm{tr}(\text{any odd number of } \gamma\text{'s}) &= 0 && (37) \\ \mathrm{tr}(\gamma^\mu\gamma^\nu) &= 4g^{\mu\nu} && (38) \\ \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) && (39) \\ & && (40) \end{aligned}$$

Another identity allows one to reverse the order of γ matrices inside a trace:

$$\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\cdots) = \mathrm{tr}(\cdots\gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu) \tag{41}$$

References

Field Theory: A Modern Primer, Pierre Ramond, Westview, 2nd edition **Chapter 8**

Introduction to Elementary Particles, D. J. Griffiths, Wiley-VCH, Second/Revised edition, 2013 **Chapters 6 & 7**

An Introduction to Quantum Field Theory, M. E. Peskin, D. V. Schroeder, ABP, 2012 **Chapter 4.8, Chapter 5.1, 5.4**

Gauge Theories in Particle Physics, I. J. R. Aitchison, A. J. G. Hey, 3rd edition, Volume 1: From Relativistic Quantum Mechanics to QED, 2011 **Chapter 8**

Quarks and Leptons: An Introductory Course in Modern Particle Physics, F. Halzen and A. D. Martin, WILEY, 1984 **Chapters 4, 5, 6**

A first book of Quantum Field Theory, A. Lahiri, P. B. Pal, Second edition, 2004, Narosa **Chapters 7, 9**