1. Four-Vector Computations

Compute the partial derivatives w.r.t. x of

$$\partial_\mu \mathrm{e}^{-ipx}\,, \quad (a\partial)(b\partial)\mathrm{e}^{-ipx}\,, \quad \partial^2 \mathrm{e}^{-ipx}\,, \quad \partial^2 \mathrm{e}^{-x^2/2}$$

for constant four vectors a, b and p, and show that

$$\partial_\mu x^\mu = 4\,.$$

2. Euler-Lagrange Formalism

The Euler-Lagrange equations can be derived analogous to classical mechanics from the action S, the time integral of the Lagrangian L via the *principle of the least action* stating the system chooses the path for which S is an extremum (usually a minimum).

(a) How are the action S and the Lagrangian density $\mathcal{L}$ for a field $\phi(x)$ defined?

(b) Derive the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = 0$$

by considering small variations to the fields $\phi(x) \to \phi'(x) = \phi(x) + \delta\phi(x)$ that vanish on the boundary of the volume.

(c) For discrete systems, the Hamiltonian (total energy of the system) is defined as the spatial integral of the Hamiltonian density

$$H = \int dx \mathcal{H} = \sum p\dot{q} - L$$

for each dynamical variable q with conjugate momentum $p = \partial\mathcal{L}/\partial\dot{q}, \dot{q} = \partial\mathcal{L}/\partial t$. Show that this can be generalized for a continuous system to

$$H = \int d^3x [\pi(\vec{x})\dot{\phi} - \mathcal{L}]$$

where $\pi(\vec{x})$ is the momentum density conjugate $\pi(\vec{x}) = \partial\mathcal{L}/\partial\dot{\phi}(\vec{x})$.

3. Klein-Gordon Equation

The kinematics of a single, scalar and real-valued field $\phi(x)$ are governed by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

where m is a constant.

(a) Derive the equations of motion using the usual procedure.

(b) What kind of system can this equation describe?

(c) What is the solution for a free scalar particle? How can the parameter m be interpreted?
(d) What is the constructed Hamiltonian?

4. Dirac Equation

The Dirac equation can also be derived by applying the Euler–Lagrange equations. Consider the Lagrangian density

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi, \quad \bar{\psi} = \psi^\dagger \gamma^0 .$$

(a) Apply Euler–Lagrange equations to $\bar{\psi}$ to derive the Dirac equation.
(b) What does it describe and what kind of objects are the ψ?
(c) Show that the Dirac equation implies the Klein–Gordon equation

$$(\partial_\mu \partial^\mu + m^2)\psi(x) = 0 .$$

5. Proca Equation

Consider the Lagrangian for a massive spin-1 particle

$$\mathcal{L} = -\frac{1}{2}\partial_\mu \phi_\nu \partial^\mu \phi^\nu + \frac{1}{2}\partial_\mu \phi^\mu \partial_\nu \phi^\nu + \frac{1}{2}m^2 \phi_\mu \phi^\mu .$$

Derive the equations of motion for ϕ^μ. Show that the field ϕ^μ fulfills the constraint $\partial_\mu \phi^\mu = 0$.