1. Second Quantization The path to quantum field theory involves roughly two steps: the change from a discrete mechanical system to fields with infinite degrees of freedom and a quantization of these fields. For the latter consider a highly simplified (spin-0, one-dimensional) Lagrangian of the field $\hat{\phi}(x,t)$

$$\hat{\mathcal{L}} = \frac{1}{2}\left(\frac{\partial\hat{\phi}}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\hat{\phi}}{\partial x}\right)^2 .$$

Following the ideas of quantum mechanics, $\hat{\phi}(x,t)$ can be expressed as a Fourier expansion of the creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$:

$$\hat{\phi}(x,t) = \int \frac{dk}{2\pi\sqrt{2\omega}}[\hat{a}(k)e^{ikx-i\omega t} + \hat{a}^\dagger a(k)e^{-ikx+i\omega t}] .$$

(a) Calculate the 'momentum field' $\hat{\pi}(x,t)$.

(b) Verify that imposing the following commutation relations for $\hat{a}$ and $\hat{a}^\dagger$

$$[\hat{a}(k), \hat{a}^\dagger(k')] = 2\pi\delta(k-k')$$
$$[\hat{a}(k), \hat{a}(k')] = [\hat{a}^\dagger(k), \hat{a}^\dagger(k')] = 0$$

are consistent with the equal time commutation relation between $\hat{\pi}$ and $\hat{\phi}$

$$\left[\hat{\phi}(x,t), \hat{\pi}(y,t)\right] = i\delta(x-y) .$$

(c) Consider the unequal time commutator $D(x_1,x_2) = \left[\hat{\phi}(\vec{x}_1,t_1), \hat{\phi}(\vec{x}_2,t_2)\right]$, where $\hat{\phi}$ is a massive KG field in three dimensions. Show that

$$D(x_1,x_2) = \int \frac{dk^3}{(2\pi)^3 2E}\left[e^{-ik\cdot(x_1-x_2)} - e^{ik\cdot(x_1-x_2)}\right]$$

where $k\cdot(x_1-x_2) = E(t_1-t_2) - \vec{k}\cdot(\vec{x}_1-\vec{x}_2)$ and $E = (\vec{k}^2+m^2)^{1/2}$. Show that $D(x_1,x_2)$ vanishes for $t_1 = t_2$ and use its Lorentz invariance to show that it vanishes for all space-like separations $(x_1-x_2)^2 < 0$.

(d) Derive the expression for the Hamiltonian $\hat{H}$.

(e) Insert the expansions of $\hat{\phi}$ and $\hat{\pi}$ into your result of (d) and verify that

$$\hat{H} = \int \frac{dk}{2\pi}\left\{\frac{1}{2}[\hat{a}^\dagger(k)\hat{a}(k) + \hat{a}(k)\hat{a}^\dagger(k)]\omega\right\}.$$

2. Dirac Spinors

The Dirac equation $(\gamma^\mu p_\mu - m)\,\psi(x) = 0$ has the following spinor solutions $u^{(1)}, u^{(2)}, v^{(1)}$ and $v^{(2)}$:

$$u^{(1)} = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m}\end{pmatrix}, \qquad u^{(2)} = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m}\end{pmatrix},$$

$$v^{(1)} = \sqrt{E+m}\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1\end{pmatrix}, \qquad v^{(2)} = -\sqrt{E+m}\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0\end{pmatrix}.$$

(a) Show that for $\vec{p} = (0, 0, p_z)$, the Dirac spinor $u^{(1)}$ takes the form

$$u^{(1)} = \begin{pmatrix} \sqrt{E+m} \\ 0 \\ \sqrt{E-m} \\ 0 \end{pmatrix} .$$

Also calculate $u^{(2)}, v^{(1)}$ and $v^{(2)}$ for this case.

(b) Show that the spinors given in part (a) are eigenstates under the spin operator S_z and determine the eigenvalues.

(c) How do the spinors in (a) look in the non-relativistic limit $(\vec{p}^2 \ll m^2)$?

(d) How do the spinors in (a) look in the ultra-relativistic limit $(\vec{p}^2 \gg m^2)$?

3. Intrinsic Parity

A Δ^0 baryon (quark content udd, spin 3/2) decays through the strong force as $\Delta^0 \to p\pi^-$.

(a) Which angular momenta of the $p\pi^-$ system are allowed by angular momentum conservation?

(b) From the decay angle distribution of the decay products an angular momentum $l = 1$ can be derived. What is thus the intrinsic parity of the Δ^0?

(c) Why can the intrinsic parity of K^+ mesons not be determined from the $K^+ \to \pi^+\pi^-$ decay? Propose a reaction with which the parity of the K^+ can be measured in principle. Draw the corresponding quark-line diagrams.

4. Pion Decay

Consider the pion decay (assume massless neutrinos):

$$\pi^+ \to e^+ + \nu_e , \quad \pi^+ \to \mu^+ + \nu_\mu .$$

(a) Draw the quark-line diagram for the decay $\pi^+ \to e^+ + \nu_e$, including the force carrier particle. Also draw the spin configuration of the decay products.

(b) Because of its finite mass, the antilepton has a small right-handed component, proportional to $1 - \beta_\ell$. The transition probability $|\mathcal{M}_{\pi\ell}|^2$ is therefore proportional to $1 - \beta_\ell$. Derive an equation for $1 - \beta_\ell$, by first deriving equations for the momentum p_ℓ and the energy E_ℓ of the charged lepton ℓ as function of the lepton mass m_ℓ and pion mass m_π.

(c) Which decay happens more often and why? What would happen if $m_e = m_\mu = 0$?

(d) Which process does one get when parity conjugation $\hat{P}$ is applied to the decay $\pi^+ \to e^+\nu_e$? Sketch the spin configuration of the decay products. Is this process allowed?