1. Solutions to the Dirac Equation

Consider once again the Dirac equation $(i\gamma^\mu\partial_\mu - m)\psi = 0$.

(a) By looking for free-particle plane wave solutions of the form $\psi = u(p)e^{-ipx}$ derive the Dirac equation for the spinor $u(p)$.

(b) How do the solutions for a particle at rest with $\vec{p} = 0$ look like? What are the negative energy solutions?

(c) How are these negative energy solutions interpreted in the Feynman-Stückelberg interpretation? Consider the $e^+e^- \to \gamma e^+e^-$ annihilation process and discuss energy and charge conservation for the two cases where

i. the negative solutions of the Dirac equation are interpreted as negative energy particles propagating backwards in time;

ii. the negative solutions of the Dirac equation are interpreted as positive energy anti-particles propagating forwards in time.

(d) Using this picture, how does the Dirac equation for antiparticle spinors look like?

(e) Starting from your result from (a) show that the corresponding result for the adjoint spinor $\bar{u} = u^\dagger\gamma^0$ is

$$\bar{u}(\gamma^\mu p_\mu - m) = 0\,.$$

From that, without using the explicit form of the u spinors, show that the normalization condition $u^\dagger u = 2E$ leads to

$$\bar{u}u = 2m$$

and that

$$\bar{u}\gamma^\mu u = 2p^\mu\,.$$

(f) Derive the identity

$$\bar{u}(p')\gamma^\mu u(p) = \frac{1}{2m}\bar{u}(p')(p+p')^\mu u(p) + \frac{i}{m}\bar{u}(p')\Sigma^{\mu\nu}q_\nu u(p)\,,$$

where $q = p' - p$ and $\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ by starting from the dirac equations for spinors and adjoint spinor.

2. Chirality Operators

(a) Show that the chirality operators $P_R = \frac{1}{2}(1+\gamma^5)$ and $P_L = \frac{1}{2}(1-\gamma^5)$ have the following properties of a projection operator:

i. $P_L^2 = P_L, P_R^2 = P_R$,

ii. $P_L + P_R = \mathbf{1}$,

iii. $P_R P_L = P_L P_R = 0$.

(b) Apply P_R and P_L to the four Dirac spinors for massless fermions (see sheet 2 with $m = 0$ and choose $\vec{p} = (0, 0, p_z)$) and interpret the result.

(c) Are the spinors still eigenvectors of the chirality operators if $m \neq 0$?

3. Chirality and Helicity

(a) Show that QED currents $\bar{\psi}^{(f)}\gamma^\mu\psi^{(i)}$ conserve chirality.

(b) Are helicity-flip reactions allowed in QED?

(c) Show that the current $\bar{\psi}^{(f)}\gamma^{\mu}\gamma^{5}\psi^{(i)}$ conserves chirality.

(d) Are helicity-flip reactions allowed in weak interactions?

(e) Show that the $V-A$ form of the antifermion current $\bar{v}\gamma^{\mu}\frac{1}{2}(1-\gamma^{5})v$ is equivalent to the statement that only the right-handed chirality components of antifermions participate in weak interactions.

References

[1] Griffiths, David J. *Introduction to Elementary Particles; 2nd rev. version*. Wiley, 2008.

[2] Thomson, Mark. *Modern Particle Physics*. Cambridge University Press, 2013.