1. Gamma Gymnastics

Prove the following identities without using a particular representation of the Dirac matrices:

(a) $\gamma^\mu \gamma_\mu = 4$

(b) $\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu$

(c) $\gamma^\mu \gamma^\nu \gamma^\lambda \gamma_\mu = 4g^{\mu\lambda}$

(d) $\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\kappa \gamma_\mu = -2\gamma^\kappa \gamma^\lambda \gamma^\nu$

(e) $\gamma^0 \gamma^{\mu\dagger} = \gamma^\mu \gamma^0$

(f) $\gamma^\mu \gamma_5 + \gamma_5 \gamma^\mu = 0$

(g) $\gamma_5 \gamma_5 = 1$

2. Fermi Theory of Neutrino-Electron Scattering

The idea of Fermi's description of the weak interactions was keep to the analogy with the well-established electromagnetic interaction but to omit the propagator term to account for the extremely small range of weak interactions. Thus he arrived at a pointlike interaction with the strength of the interaction described by a constant G_F which was assumed to be universal for all processes of weak interactions.

Consider the scattering of muon neutrinos on electrons, where a muon together with an electron neutrino can be created.

(a) In the Fermi theory the lowest-order total cross section for this process is

$$\sigma = \frac{G_F^2}{\pi} \cdot s$$

with the center-of-mass energy $\sqrt{s} = 2|\vec{p}^*|$, where $\vec{p}^*$ is the momentum of one of the fermions in the center-of-mass system. How large is the cross section (in barn) for the scattering of the muon neutrinos with an energy $E_{\nu_\mu} = 10\,$GeV (use $G_F = 1.166 \cdot 10^{-5}\mathrm{GeV}^{-2}$ in natural units)?

(b) As the description assumes a point-like interaction, the following unitarity bound

$$\frac{G_F^2}{\pi} s \leq \frac{4\pi}{s}$$

holds. At which neutrino energy and which cross section would the bound be violated?

(c) In the electroweak theory the process above is described through the exchange of a W boson rather via point-like interactions. The cross section can be written as

$$\sigma = \frac{G_F^2}{\pi} \cdot s \cdot \frac{M_W^2}{s + M_W^2} \, .$$

Draw the Feynman diagram and calculate the maximum possible cross section.

3. Interconnectivity of e, g and $g^{'}$

Consider the covariant derivative D_μ of the $SU(2)_L \otimes U(1)_Y$ for leptons.

(a) Show that D_μ can be written for right-handed electrons as

$$D_\mu^R = \partial_\mu - ig' \cos\theta_w A_\mu(x) + ig' \sin\theta_w Z_\mu(x)$$

(b) Show that D_μ can be written for left-handed electrons as

$$D_\mu^L = \partial_\mu + i\frac{g}{\sqrt{2}}(\tau_+ W_\mu^+(x) + \tau_- W_\mu^-(x)) - i\frac{g\sin\theta_w + g'\cos\theta_w}{2} A_\mu(x) + i\frac{-g\cos\theta_w + g'\sin\theta_w}{2} Z_\mu(x)$$

where τ_+ and τ_- are the isospin raising and lowering operators:

$$\tau_+ = \frac{1}{2}(\tau_1 + i\tau 2) \quad \tau_- = \frac{1}{2}(\tau_1 - i\tau 2)$$

with Pauli matrices τ_i.

(c) With the previous results show that

$$e = g'\cos\theta_w = g\sin\theta_w.$$

In a similar fashion show that the absence of a neutrino-photon coupling leads to

$$g'\cos\theta_w = g\sin\theta_w.$$

4. Representations of SU(2)

The group SU(2) corresponds to the set of special unitary transformations which act on complex 2D vectors under the operation of matrix multiplication. The natural representation is that of 2x2 matrices acting on 2D vectors. The group has $2^2 - 1 = 3$ traceless, hermitian generators labeled as J_1, J_2 and J_3.

(a) Show that SU(n) indeed fulfills the definitions of a group G, i.e.

  i. Closure: $\forall a, b \in G, a \cdot b \in G$
  ii. Associativity: $\forall a, b, c \in G, (a \cdot b) \cdot c = a \cdot (b \cdot c)$
  iii. Identity: existence of a neutral element e so that $a \cdot e = a$
  iv. Inverse: existence of an inverse element a^{-1} so that $a \cdot a^{-1} = e$

(b) Form a 2D representation of SU(2), i.e. J_1, J_2 and J_3. by choosing the following two orthogonal states as base vectors

$$\begin{pmatrix}1\\0\end{pmatrix} \equiv \left|\frac{1}{2}, +\frac{1}{2}\right\rangle \quad \text{and} \quad \begin{pmatrix}0\\1\end{pmatrix} \equiv \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$$

(c) Form an alternative 2D representation of SU(2) by choosing the following two orthogonal states as base vectors

$$\begin{pmatrix}1\\0\end{pmatrix} \equiv \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}, +\frac{1}{2}\right\rangle + \left|\frac{1}{2}, -\frac{1}{2}\right\rangle\right) \quad \text{and} \quad \begin{pmatrix}0\\1\end{pmatrix} \equiv \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}, +\frac{1}{2}\right\rangle - \left|\frac{1}{2}, -\frac{1}{2}\right\rangle\right)$$

References

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[3] Haywood, Stephen. *Symmetries and Conservation Laws in Particle Physics*. Imperial College Press, 2011.