1. Second Quantization

The path to quantum field theory involves roughly two steps: the change from a discrete mechanical system to fields with infinite degrees of freedom and a quantization of these fields. For the latter consider a highly simplified (spin-0, one-dimensional) Lagrangian of the field $\hat{\phi}(x,t)$

$$\hat{\mathcal{L}} = \frac{1}{2}\left(\frac{\partial\hat{\phi}}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\hat{\phi}}{\partial x}\right)^2 .$$

Following the ideas of quantum mechanics, $\hat{\phi}(x,t)$ can be expressed as a Fourier expansion of the creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$:

$$\hat{\phi}(x,t) = \int \frac{dk}{2\pi\sqrt{2\omega}}[\hat{a}(k)e^{ikx-i\omega t} + \hat{a}^\dagger a(k)e^{-ikx+i\omega t}] .$$

(a) Calculate the 'momentum field' $\hat{\pi}(x,t)$.

(b) Verify that imposing the following commutation relations for $\hat{a}$ and $\hat{a}^\dagger$

$$[\hat{a}(k), \hat{a}^\dagger(k')] = 2\pi\delta(k-k')$$
$$[\hat{a}(k), \hat{a}(k')] = [\hat{a}^\dagger(k), \hat{a}^\dagger(k')] = 0$$

are consistent with the equal time commutation relation between $\hat{\pi}$ and $\hat{\phi}$

$$\left[\hat{\phi}(x,t), \hat{\pi}(y,t)\right] = i\delta(x-y) .$$

(c) Derive the expression for the Hamiltonian $\hat{H}$.

(d) You can verify that

$$\hat{H} = \int \frac{dk}{2\pi}\left\{\frac{1}{2}[\hat{a}^\dagger(k)\hat{a}(k) + \hat{a}(k)\hat{a}^\dagger(k)]\omega\right\}.$$

by inserting the expansions of $\hat{\phi}$ and $\hat{\pi}$ into your result of (c). How can we thus interpret the $\hat{a}(k)$ and $\hat{a}^\dagger(k)$?

2. Solutions to the Dirac Equation

Consider once again the Dirac equation $(i\gamma^\mu\partial_\mu - m)\psi = 0$.

(a) By looking for free-particle plane wave solutions of the form $\psi = u(p)e^{-ipx}$ derive the Dirac equation for the spinor $u(p)$.

(b) How do the solutions for a particle at rest with $\vec{p} = 0$ look like? What are the negative energy solutions?

(c) How are these negative energy solutions interpreted in the Feynman-Stückelberg interpretation? Consider the $e^+e^- \to \gamma e^+e^-$ annihilation process and discuss energy and charge conservation for the two cases where

i. the negative solutions of the Dirac equation are interpreted as negative energy particles propagating backwards in time;

ii. the negative solutions of the Dirac equation are interpreted as positive energy antiparticles propagating forwards in time.

(d) Using this picture, how does the Dirac equation for antiparticle spinors look like?

(e) Starting from your result from (a) show that the corresponding result for the adjoint spinor $\bar{u} = u^\dagger \gamma^0$ is

$$\bar{u}(\gamma^\mu p_\mu - m) = 0\,.$$

From that, without using the explicit form of the u spinors, show that the normalization condition $u^\dagger u = 2E$ leads to

$$\bar{u}u = 2m$$

and that

$$\bar{u}\gamma^\mu u = 2p^\mu\,.$$

3. Representations of SU(2)

The group SU(2) corresponds to the set of special unitary transformations which act on complex 2D vectors under the operation of matrix multiplication. The natural representation is that of 2x2 matrices acting on 2D vectors. The group has $2^2 - 1 = 3$ traceless, hermitian generators labeled as J_1, J_2 and J_3.

(a) Show that SU(n) indeed fulfills the definitions of a group G, i.e.

i. Closure: $\forall a, b \in G, a \cdot b \in G$

ii. Associativity: $\forall a, b, c \in G, (a \cdot b) \cdot c = a \cdot (b \cdot c)$

iii. Identity: existence of a neutral element e so that $a \cdot e = a$

iv. Inverse: existence of an inverse element a^{-1} so that $a \cdot a^{-1} = e$

(b) Form a 2D representation of SU(2), i.e. J_1, J_2 and J_3. by choosing the following two orthogonal states as base vectors

$$\begin{pmatrix}1\\0\end{pmatrix} \equiv \left|\frac{1}{2}, +\frac{1}{2}\right\rangle \quad \text{and} \quad \begin{pmatrix}0\\1\end{pmatrix} \equiv \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$$

(c) Form an alternative 2D representation of SU(2) by choosing the following two orthogonal states as base vectors

$$\begin{pmatrix}1\\0\end{pmatrix} \equiv \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}, +\frac{1}{2}\right\rangle + \left|\frac{1}{2}, -\frac{1}{2}\right\rangle\right) \quad \text{and} \quad \begin{pmatrix}0\\1\end{pmatrix} \equiv \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}, +\frac{1}{2}\right\rangle - \left|\frac{1}{2}, -\frac{1}{2}\right\rangle\right)$$