1. Gamma Gymnastics

Prove the following identities without using a particular representation of the Dirac matrices:

(a) $\gamma^\mu \gamma_\mu = 4$
(b) $\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu$
(c) $\gamma^\mu \gamma^\nu \gamma^\lambda \gamma_\mu = 4g^{\mu\lambda}$
(d) $\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\kappa \gamma_\mu = -2\gamma^\kappa \gamma^\lambda \gamma^\nu$
(e) $\gamma^0 \gamma^{\mu\dagger} = \gamma^\mu \gamma^0$
(f) $\gamma^\mu \gamma_5 + \gamma_5 \gamma^\mu = 0$
(g) $\gamma_5 \gamma_5 = 1$

2. Spin of a Two-Nucleon System

The isopin states $|I, I_3\rangle$ of a two-nucleon system can be obtained as a superposition of single particle states involving the proton (p) with $|I_p, I_{3p}\rangle = |\frac{1}{2}, +\frac{1}{2}\rangle$ and the neutron (n) with $|I_n, I_{3n}\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle$.

(a) Write down the four isopin states of a two-nucleon system.
(b) Which of these states is the deuteron?
(c) From measurements, it is known the the relative orbital angular momentum of p and n is $L = 0$ with an admixture of $L = 2$ (i.e., the spatial part of the wavefunction is even under the exchange of p and n). Give an argument why the spin of the deuteron is $S = 1$.

3. Pion Decay

Consider the pion decay (assume massless neutrinos):

$$\pi^+ \to e^+ + \nu_e\,, \quad \pi^+ \to \mu^+ + \nu_\mu\,.$$

(a) Draw the quark-line diagram for the decay $\pi^+ \to e^+ + \nu_e$, including the force carrier particle. Also draw the spin configuration of the decay products.
(b) Because of its finite mass, the antilepton has a small right-handed component, proportional to $1 - \beta_\ell$. The transition probability $|\mathcal{M}_{\pi\ell}|^2$ is therefore proportional to $1 - \beta_\ell$. Derive an equation for $1 - \beta_\ell$, by first deriving equations for the momentum p_ℓ and the energy E_ℓ of the charged lepton ℓ as function of the lepton mass m_ℓ and pion mass m_π.
(c) Which decay happens more often and why? What would happen if $m_e = m_\mu = 0$?
(d) Which process does one get when parity conjugation $\hat{P}$ is applied to the decay $\pi^+ \to e^+ \nu_e$? Sketch the spin configuration of the decay products. Is this process allowed?

4. Interconnectivity of e, g and $g^{'}$

Consider the covariant derivative D_μ of the $SU(2)_L \otimes U(1)_Y$ for leptons.

(a) Show that D_μ can be written for right-handed electrons as

$$D^R_\mu = \partial_\mu - ig' \cos\theta_w A_\mu(x) + ig' \sin\theta_w Z_\mu(x)$$

(b) Show that D_μ can be written for left-handed electrons as

$$D_\mu^L = \partial_\mu + i\frac{g}{\sqrt{2}}\big(\tau_+ W_\mu^+(x) + \tau_- W_\mu^-(x)\big) - i\frac{g\sin\theta_w + g'\cos\theta_w}{2}A_\mu(x)$$
$$+ i\frac{-g\cos\theta_w + g'\sin\theta_w}{2}Z_\mu(x)$$

where τ_+ and τ_- are the isospin raising and lowering operators:

$$\tau_+ = \frac{1}{2}(\tau_1 + i\tau 2) \quad \tau_- = \frac{1}{2}(\tau_1 - i\tau 2)$$

with Pauli matrices τ_i.

(c) With the previous results show that

$$e = g'\cos\theta_w = g\sin\theta_w.$$

In a similar fashion show that the absence of a neutrino-photon coupling leads to

$$g'\cos\theta_w = g\sin\theta_w.$$

References

[1] Aitchison, Ian. *Gauge Theories in Particle Physics Volume 1*. CRC Press, 2013.

[2] Griffiths, David J. *Introduction to Elementary Particles; 2nd rev. version*. Wiley, 2008.

[3] Thomson, Mark. *Modern Particle Physics*. Cambridge University Press, 2013.