1. Lagrange Density for the Higgs Field

Consider the Lagrangian for a complex scalar field ϕ

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi)^* - V(\phi)$$

with the potential $V(\phi) = \alpha|\phi|^2 + \beta|\phi|^4$ where α and β are real coefficients.

(a) Show that the Lagrangian is invariant under a global $U(1)$ transformation but not under a local $U(1)$ transformation.

(b) Which condition must α and β fulfill so that spontaneous symmetry breaking occurs?

(c) What happens if $\beta < 0$?

2. The W and Z Boson Mass

The simplest Higgs field that can give mass to the gauge bosons of the weak interaction consists of complex fields ϕ^+ and ϕ^0, which form a weak-isospin doublet ($I = 1/2$, weak hypercharge $Y = 1$):

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}.$$

The Lagrangian of the free Higgs field and the Higgs-gauge boson coupling is given by

$$\mathcal{L} = (D_\mu \Phi)^\dagger (D^\mu \Phi) - V(\Phi^\dagger \Phi) \quad \text{with} \quad V(\Phi^\dagger \Phi) = -\mu^2 \Phi^\dagger \Phi + \lambda^2 (\Phi^\dagger \Phi)^2$$

with the covariant derivative

$$D_\mu = \partial_\mu + \frac{1}{2} i g' Y B_\mu(x) + \frac{1}{2} i g \vec{\sigma} \vec{W}_\mu(x).$$

The $SU(2)_L \otimes U(1)_Y$ symmetry is spontaneously broken by the ground state.

(a) The ground state of the Higgs field Φ_0 can be chosen to be

$$\Phi_0 = \begin{pmatrix} \phi_0^+ \\ \phi_0^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix},$$

where v is, without loss of generality, a real parameter. Assume

$$A_\mu = B_\mu \cos\theta_w + W_\mu^3 \sin\theta_w$$
$$Z_\mu = -B_\mu \sin\theta_w + W_\mu^3 \cos\theta_w$$

and show that $\phi_0^+ \neq 0$ would lead to a coupling of the Higgs field to the photon. Also show that such a coupling does not appear if $\phi_0^0 \neq 0$.

(b) Expand the Higgs field $\Phi(x)$ around the ground state considering small perturbations $\eta(x)$ around the vacuum expectation value

$$\Phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + \eta(x) \end{pmatrix},$$

as in the lecture (the $i\zeta(x)$ term can be ignored, why?) to show that the terms in the Lagrange density describing the mass of the electroweak gauge bosons and the Higgs boson can be written as

$$\mathcal{L} = \left[\frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - \mu^2 \eta^2\right] + \frac{1}{2}\frac{g^2 v^2}{4}\left[|W_\mu^+|^2 + |W_\mu^-|^2\right] + \frac{1}{2}\frac{g^2 v^2}{4\cos^2\theta_w}|Z_\mu|^2.$$

(c) What do you find for the masses of the Higgs boson and the electroweak gauge bosons?

3. The HZZ Interaction

By considering the interactions terms of the Higgs field with the gauge bosons originating from $(D_\mu\phi)^\dagger(D^\mu\phi)$, show that the HZZ coupling is given by

$$g_{HZZ} = \frac{g_W}{\cos\theta_W} m_Z \,.$$

References

[1] Aitchison, Ian. *Gauge Theories in Particle Physics Volume 1*. CRC Press, 2013.

[2] Griffiths, David J. *Introduction to Elementary Particles; 2nd rev. version*. Wiley, 2008.

[3] Thomson, Mark. *Modern Particle Physics*. Cambridge University Press, 2013.