Testing the Standard Model of Elementary Particle Physics I

Second lecture

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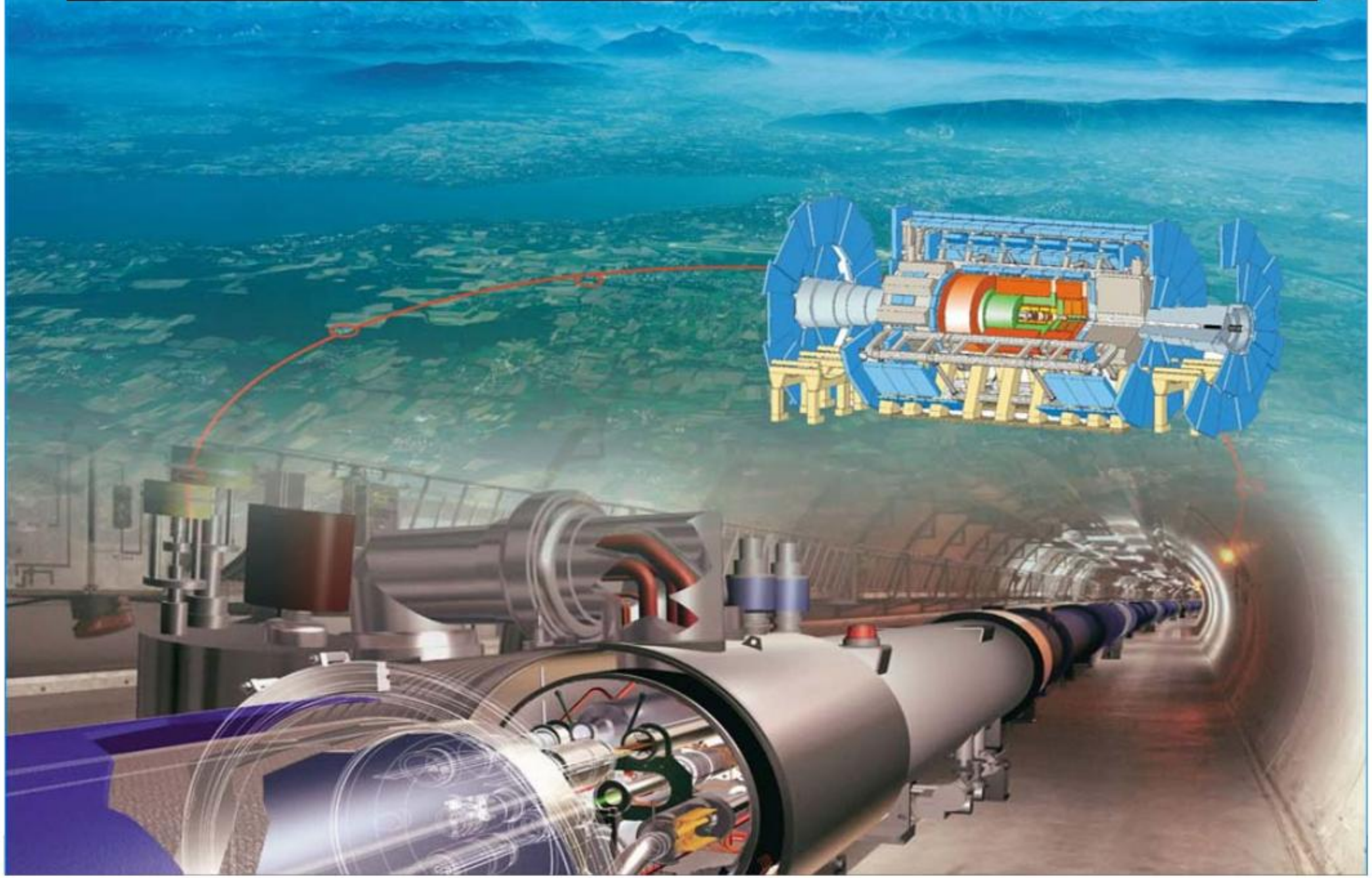
1.2 Relativistic quantum mechanics



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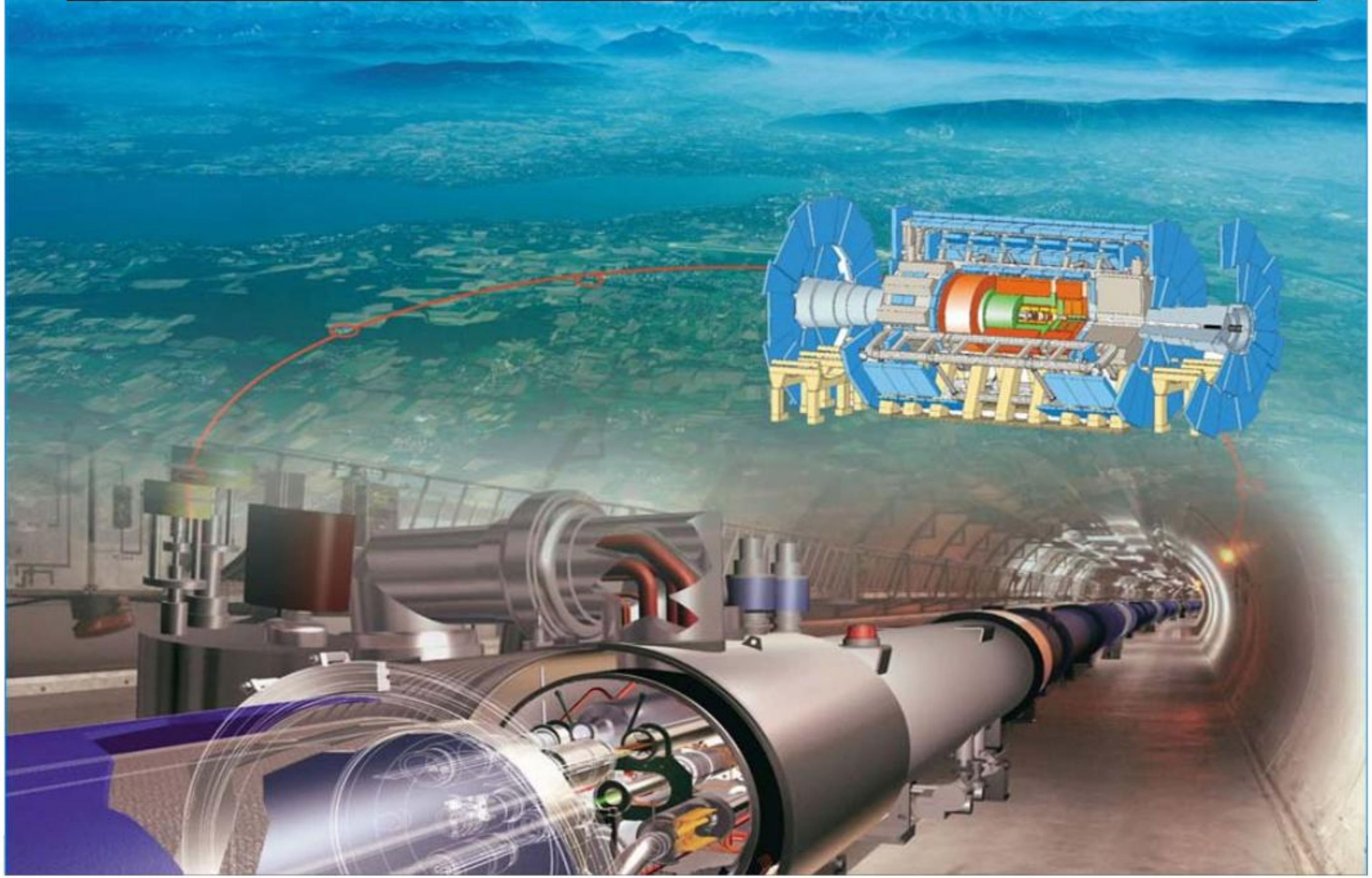
1.2.1 Reminder of matrix algebra



Reminder: Matrix algebra

- **The trace of a matrix A:** $\mathrm{Tr}(A) = \sum_{i}^{n} a_{ii}$

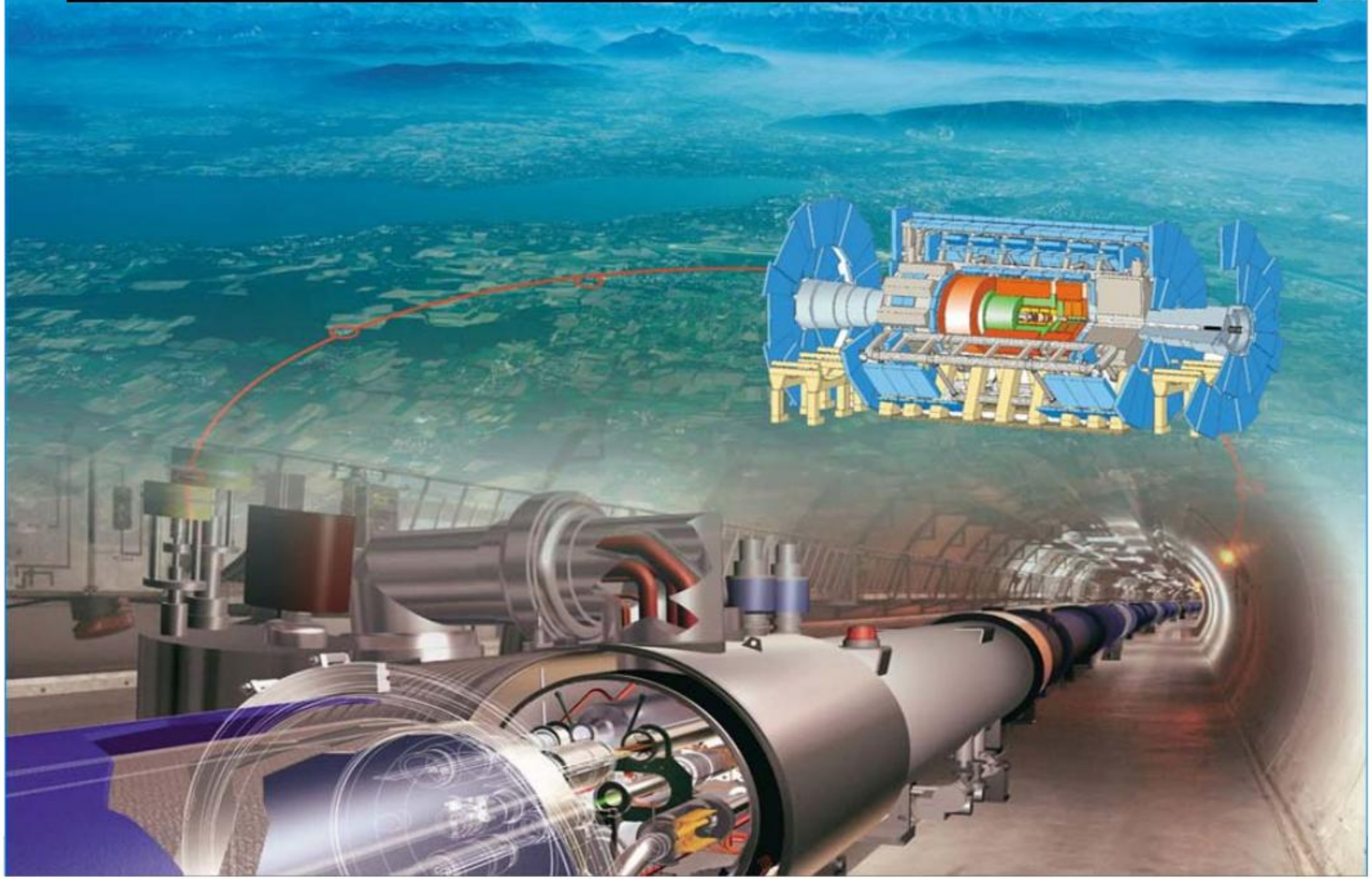
- **Hermitian conjugate of A** is obtained by taking the transpose and then the complex conjugate:

$$A \to A^{\dagger} = \bar{A}^{T} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{T} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

- **Commuting matrices:** $[A, B] = AB - BA = 0$

- **Non-commuting matrices:** $[A, B] = AB - BA \neq 0$

1.2.2 Notations of special relativity



Notations of special relativity

- **In special relativity, the components of a four-vector x^μ are defined by three spatial coordinates and time.**
  - Greek letters μ, ν, λ, ... will be used to indicate components of a four-vector
  - Latin indices i, j, k, ... will be used to indicate its spatial components
- **Contravariant coordinates:**

$$x^\mu \equiv (x^0, x^1, x^2, x^3) \equiv (t, \vec{x})$$

- **Covariant coordinates:**

$$x_\mu \equiv (x_0, x_1, x_2, x_3) \equiv (t, -\vec{x})$$

- **Lower/raise the indices via:**

$$x_\mu = g_{\mu\nu} x^\nu = \sum_\nu g_{\mu\nu} x^\nu \text{ and } x^\mu = g^{\mu\nu} x_\nu = \sum_\nu g^{\mu\nu} x_\nu$$

Notations of special relativity

- **The Minkowski metric tensor $g_{\mu\nu}$ is defined via:**

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \equiv g^{\mu\nu}$$

- **The distance l between two points x and y in space-time can be expressed using the Minkowski metric in terms of:**

$$l^2 = g_{\mu\nu}\left(x^\mu - y^\mu\right)\left(x^\nu - y^\nu\right)$$

Notations of special relativity

- **The scalar product of two vectors A^μ and B^μ will be written as:**

$$g_{\mu\nu}A^\mu B^\nu = A_\nu B^\nu = A^\nu B_\nu = A_0 B^0 - \vec{A}\vec{B}$$

- **Covariant and contravariant derivatives are defined via**

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$$

and

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)$$

- **The wave operator (D'Alembert operator) is defined via:**

$$\Box \equiv \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 = \partial_\mu \partial^\mu$$

Notations of special relativity

- The momentum four-vector of a particle is given by:

$$p^\mu = (E, \vec{p}) \quad \text{and} \quad p_\mu = (E, -\vec{p})$$

where E is the energy of the particle.

- The invariant scalar product of the covariant and contravariant momenta is denoted by:

$$p^\mu p_\mu = E^2 - \vec{p}^2 = m^2$$

where m is the rest-mass of the particle

Notations of special relativity

- **Natural units:**
  - In discussions on **relativistic quantum mechanics (and quantum field theories)**, it is customary to use a system of units in which there is only one fundamental unit: i.e. **the unit of mass**
    - The units of length and time are defined by declaring: $\hbar = 1, \quad c = 1.$

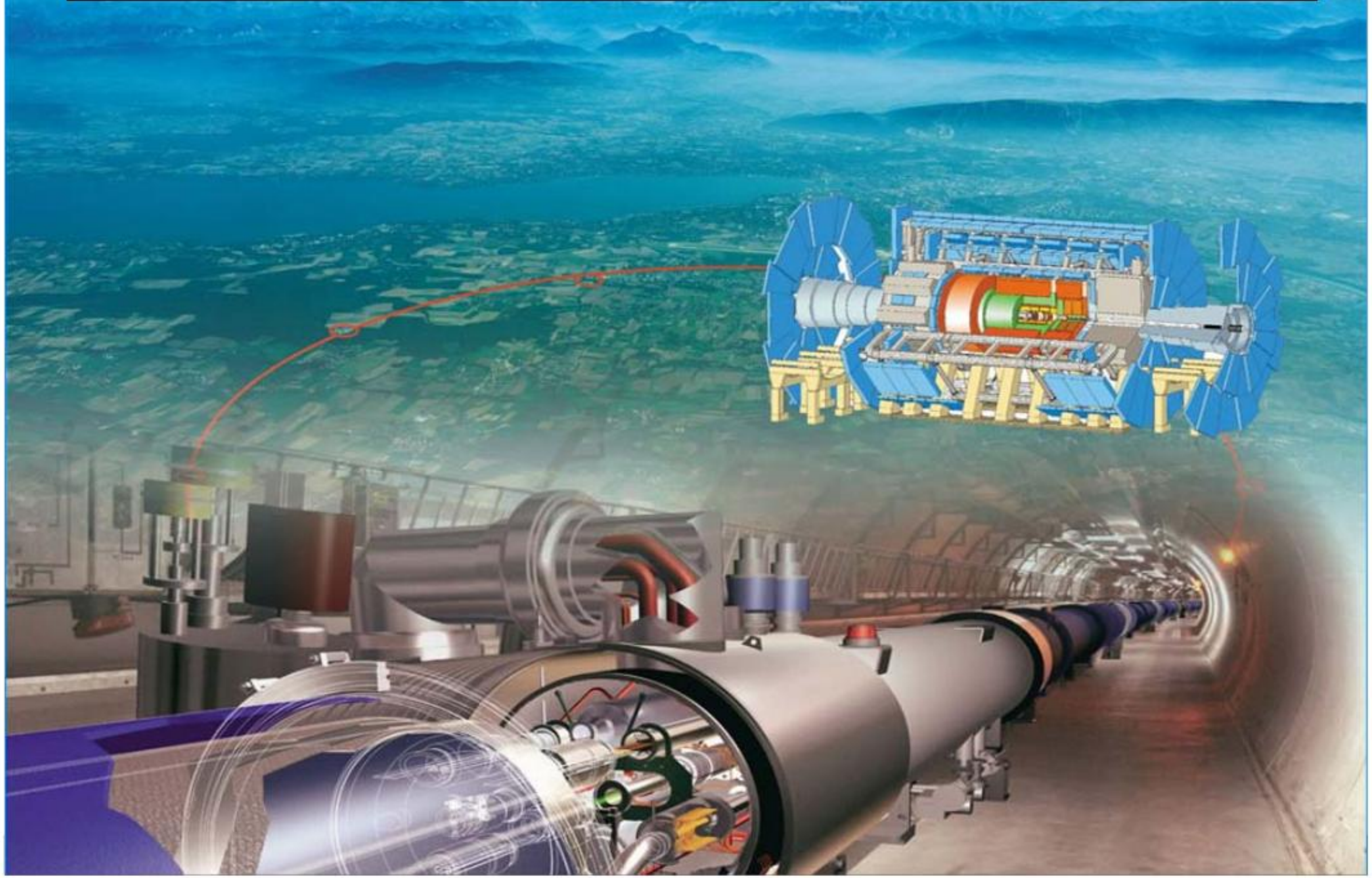
Quantity	Dimension	Conversion factor
Mass	$[M]$	$1/c^2$
Length	$[M]^{-1}$	$\hbar c$
Time	$[M]^{-1}$	$\hbar$
Energy	$[M]$	1
Momentum	$[M]$	$1/c$
Electric charge	$[M]^0$	$\sqrt{\hbar c}$

speed of light

reduced planck constant

To go from energy units (MeV or GeV) to conventional units we need to multiply by a conversion factor

1.2.3 Relativistic wavefunctions



Relativistic wavefunctions

- **Duality between matter and radiation is a striking characteristic of non-classical physics**
  - Particle-like behaviour of light (photons)
  - Wave-like behaviour of electrons:

$$\Psi\left(\vec{r}, t\right) = \frac{1}{\sqrt{V}} \exp\left(i\vec{k}\vec{r} - i\omega t\right)$$

where energy and momentum are defined via:

$$E = \hbar\omega \quad , \quad \vec{p} = \hbar\vec{k} \quad (\text{with } k = 2\pi/\lambda)$$

- **In quantum theories, quantities are represented by operators which act on the wavefunctions (their eigenvalues are measurable):**

$$E \longrightarrow i\hbar\frac{\partial}{\partial t} \quad , \quad \vec{p} \longrightarrow -i\hbar\vec{\nabla}$$

Relativistic wavefunctions

- **The Schrödinger equation is obtained after inserting the energy and momentum operators into the non-relativistic representation of the total energy:**

$$E = \frac{\vec{p}^2}{2m} + V(\vec{r}, t)$$

$$\downarrow$$

$$i\hbar \frac{\partial \Psi(\vec{r}, t)}{\partial t} = \left(-\frac{\hbar^2}{2m} \vec{\nabla}^2 + V(\vec{r}, t) \right) \Psi(\vec{r}, t) \equiv H \Psi(\vec{r}, t)$$

- **The wavefunctions gain their meaning in the context of the probability density and probability current density:**

$$\rho = \Psi^* \Psi = |\Psi|^2 \quad \text{and} \quad \vec{j} = \frac{\hbar}{2im} \left(\Psi^* \left(\vec{\nabla} \Psi \right) - \left(\vec{\nabla} \Psi^* \right) \Psi \right)$$

Relativistic wavefunctions

- The probability density and probability current density follow the continuity equation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0$$

→ probability is conserved.

- For a free particle we obtain:

$$\rho = \frac{1}{V} \quad , \quad \vec{j} = \rho \vec{v}$$

i.e. the current density is the product of the probability density and the velocity

Klein-Gordon equation

- **The Klein-Gordon equation describes relativistic scalars ($\pi^{\pm}$, K^0, Higgs boson)**
  - It is obtained after inserting the energy and momentum operators into the relativistic representation of the total energy of a free particle:

from now on:

$$\hbar = 1, \quad c = 1$$

$$E^2 = \vec{p}^2 + m^2$$

$$E \longrightarrow i\frac{\partial}{\partial t}$$

$$\vec{p} \longrightarrow -i\vec{\nabla}$$

$$-\frac{\partial^2 \Phi}{\partial t^2} = \left(-\vec{\nabla}^2 + m^2\right)\Phi$$

- **After sorting the terms we obtain the Klein-Gordon equation for a free particle with mass m:**

$$\left[\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2\right]\Phi(\vec{r},t) \equiv \left(\Box + m^2\right)\Phi(\vec{r},t) = 0$$

Klein-Gordon equation

- **The same equation is also valid for complex conjugated wavefunctions (i.e. antiparticles):**

$$\left(\Box + m^2\right) \Phi^* (\vec{r}, t) = 0$$

- **This equation is the relativistic generalisation of the Schrödinger equation**
- **Solutions are given by plane waves like:**

$$\Phi\left(\vec{r}, t\right) = \frac{1}{\sqrt{V}} \exp\left(i(\vec{k}\vec{r} \pm \omega t)\right)$$

- **The energy eigenvalues (obtained after including the wavefunctions into the Klein-Gordon function) are:**

$$E = \pm\omega = \pm\sqrt{\vec{p}^2 + m^2}$$

Klein-Gordon equation

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$$E = \pm\omega = \pm\sqrt{\vec{p}^2 + m^2}$$

Eigenvalues can be either positive or negative

Klein-Gordon equation

- **Wavefunctions with positive solutions are given via:**

$$\Phi_+(\vec{r},t) = \frac{1}{\sqrt{V}} \exp\left(i\vec{k}\vec{r} - i\omega t\right), \quad i\frac{\partial \Phi_+}{\partial t} = +\omega \Phi_+$$

- **Wavefunctions with negative solutions are given via:**

$$\Phi_-(\vec{r},t) = \frac{1}{\sqrt{V}} \exp\left(i\vec{k}\vec{r} + i\omega t\right), \quad i\frac{\partial \Phi_-}{\partial t} = -\omega \Phi_-$$

- **Negative values appeared unphysical at first.**
  - Wavefunctions with negative energy can not be ignored (as the solutions with E > 0 do not give a complete system of eigenfunctions)
    - See example from classical wave equation

Ignoring this term omits waves going to the left as well as standing waves

$$f(x,t) = a\exp(ikx - i\omega t) + b\exp(ikx + i\omega t)$$

Klein-Gordon equation

- **Wavefunctions with positive solutions are given via:**

$$\Phi_{+}(\vec{r}, t) = \frac{1}{\sqrt{V}} \exp\left(i\vec{k}\vec{r} - i\omega t\right), \quad i\frac{\partial \Phi_{+}}{\partial t} = +\omega \Phi_{+}$$

Identified as wavefunction for antiparticles

- **Wavefunctions with negative solutions are given via:**

$$\Phi_{-}(\vec{r}, t) = \frac{1}{\sqrt{V}} \exp\left(i\vec{k}\vec{r} + i\omega t\right), \quad i\frac{\partial \Phi_{-}}{\partial t} = -\omega \Phi_{-}$$

- **Negative values appeared unphysical at first.**
  - Wavefunctions with negative energy can not be ignored (as the solutions with E > 0 do not give a complete system of eigenfunctions)
    - See example from classical wave equation

Ignoring this term omits waves going to the left as well as standing waves

$$f(x, t) = a \exp(ikx - i\omega t) + b \exp(ikx + i\omega t)$$

Klein-Gordon equation

- **The probability density and probability current density for a scalar particle with positive energy are:**

$$\rho = \frac{1}{V}\frac{\omega}{m}\,, \quad j = \frac{1}{V}\frac{\vec{k}}{m}$$

- **Solutions with positive energy correspond to a positive probability density, while solutions with negative energy correspond to a negative probability density**

Dirac equation

- **The Dirac equation describes relativistic fermions (spin-½ particles):**
  - Developed by Paul Dirac (1928), who was searching for an equation that
    a) is of first-order in time to avoid negative energy solutions (as Schrödinger equation)
    b) follows the laws of special relativity
  - Dirac chose the following ansatz to describe the wavefunctions of a free electron:

$$i\frac{\partial\Psi}{\partial t} \equiv H\Psi = (\vec{\alpha}\cdot\vec{p} + \beta m)\,\Psi$$
$$= \left(-i\vec{\alpha}\vec{\nabla} + \beta m\right)\Psi$$
$$= -i\left(\alpha_1\frac{\partial\Psi}{\partial x_1} + \alpha_2\frac{\partial\Psi}{\partial x_2} + \alpha_3\frac{\partial\Psi}{\partial x_3}\right) + \beta m\Psi$$

- **The parameter α_1, α_2, α_3, and β have to be chosen such that the relativistic relationship between energy and momentum is satisfied:** $E^2 = \vec{p}^2 + m^2$

Dirac equation

- **Requirement can be achieved if α_1, α_2, α_3, and β are hermitian matrices and follow:**

1) $$\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = \beta = \mathbb{1}\ , \quad \alpha_j \alpha_k + \alpha_k \alpha_j = 0 \quad \text{for } j \neq k \quad \text{and} \quad \alpha_j \beta + \beta \alpha_j = 0.$$

eigenvalues are ±1

2) $$\mathrm{Tr}\left(\alpha_j\right) = \mathrm{Tr}\left(\beta\right) = 0$$

Thus the dimension of the matrices has to be even. However, dimension N = 2 is not sufficient because there are only three linear independent hermitian matrices with a trace of 0 (i.e. the Pauli matrices):

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Dirac equation

For N = 4 there are 16 linear independent hermitian matrices. For our problem the matrices

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$\alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

are ideal. The Dirac equation can be formulated in a spacetime symmetric way if one introduces the Gamma matrices:

$$\gamma^0 = \beta, \quad \gamma^1 = \beta\alpha_1, \quad \gamma^2 = \beta\alpha_2, \quad \gamma^3 = \beta\alpha_3$$

Dirac equation

For the sake of simplicity, the Gamma matrices are written as a 4-vector:

$$\gamma^\mu = (\gamma^0,\ \gamma^1,\ \gamma^2,\ \gamma^3)$$

With:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \cdot \mathbb{1}$$

$$(\gamma^0)^\dagger = \gamma^0\ , \quad (\gamma^j)^\dagger = -\gamma^j\ , \quad (\text{for } j = 1, 2, 3)$$

Thus the Dirac equation can be written as:

$$(i\gamma^\mu\partial_\mu - m)\,\Psi(x) = 0$$

Dirac equation

- **The solutions to the Dirac equation are referred to as Dirac Spinors:**

$$\Psi_+(x) \equiv u_{1,2}(p)\exp(-iEt)\exp(+i\vec{p}\cdot\vec{x})$$
$$\Psi_-(x) \equiv v_{1,2}(p)\exp(+iEt)\exp(-i\vec{p}\cdot\vec{x})$$

with the components:

$$\underbrace{u_1(p) = \sqrt{\frac{E+m}{V}}\cdot\begin{pmatrix}1\\0\\\frac{p_z}{E+m}\\\frac{p_x+ip_y}{E+m}\end{pmatrix}}_{\Uparrow} \qquad \underbrace{u_2(p) = \sqrt{\frac{E+m}{V}}\cdot\begin{pmatrix}0\\1\\\frac{p_x-ip_y}{E+m}\\\frac{-p_z}{E+m}\end{pmatrix}}_{\Downarrow}$$

Spinors of positive energy

$$\underbrace{v_1(p) = \sqrt{\frac{E+m}{V}}\cdot\begin{pmatrix}\frac{p_x-ip_y}{E+m}\\\frac{-p_z}{E+m}\\0\\1\end{pmatrix}}_{\Uparrow} \qquad \underbrace{v_2(p) = \sqrt{\frac{E+m}{V}}\cdot\begin{pmatrix}\frac{p_z}{E+m}\\\frac{p_x+ip_y}{E+m}\\1\\0\end{pmatrix}}_{\Downarrow}$$

Spinors of negative energy

Dirac equation

- **The solutions to the Dirac equation are referred to as Dirac Spinors:**

$$\Psi_+(x) \equiv u_{1,2}(p)\exp(-iEt)\exp(+i\vec{p}\cdot\vec{x})$$
$$\Psi_-(x) \equiv v_{1,2}(p)\exp(+iEt)\exp(-i\vec{p}\cdot\vec{x})$$

with the components:

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Spinors of positive energy

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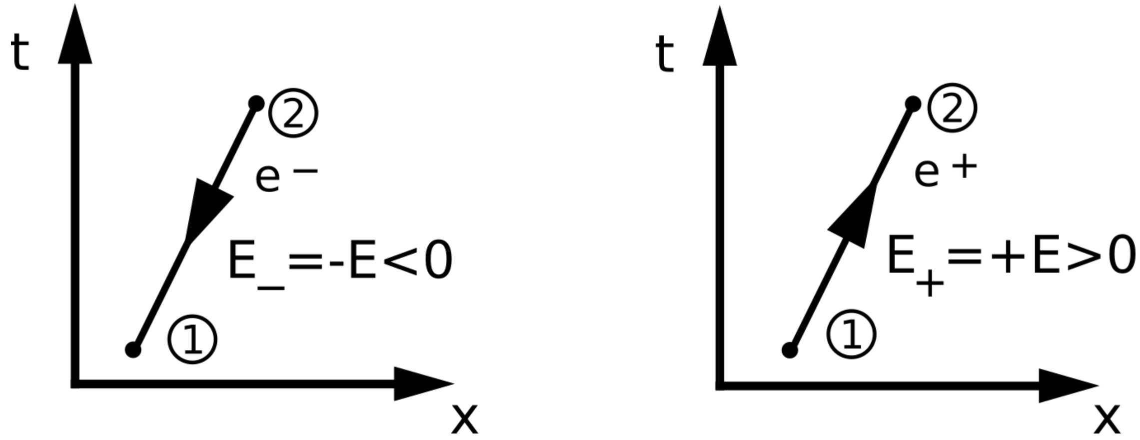
Spinors of negative energy

Representation of the direction of the fermion spin

Dirac equation

- **Negative energy solutions:**
  - **Feynman-Stückelberg interpretation:**
    - Wavefunctions of negative energies describe (for t → -t) antiparticles moving forward in time.
      1. Emission of an antiparticle with 4-momentum p^μ is equivalent to absorbing a particle with the 4-momentum $-p^\mu$
      2. Absorbing an antiparticle with 4-momentum p^μ is equivalent to emission of a particle with the 4-momentum $-p^\mu$



Dirac equation

- **Interpretation of spinors:**
  - $u_{1,2}(p)$ incoming fermion annihilated at interaction point (E > 0)
  - $\bar{u}_{1,2}(p)$ outgoing fermion created at interaction point (E > 0)
  - $v_{1,2}(p)$ incoming antifermion created at interaction point (E < 0)
  - $\bar{v}_{1,2}(p)$ outgoing antifermion annihilated at interaction point (E < 0)
- **Dirac adjoint spinor is defined as:**

$$\bar{\Psi} = \Psi^{\dagger}\gamma^{0}$$

and follows the adjoint Dirac equation:

$$\bar{\Psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right) = 0$$

- **The probability density is defined via:**

$$\rho = \Psi^{\dagger}\Psi = |\Psi_1|^2 + |\Psi_2|^2 + |\Psi_3|^2 + |\Psi_4|^2$$

Dirac equation

- **Helicity and chirality:**
  - **Helicity** is defined as the projection of the spin orientation onto the direction of the momentum:

$$\lambda = \frac{\vec{s} \cdot \vec{p}}{|\vec{p}|}$$

    - **Fermions** with λ = +½ have parallel spin and momentum
    - **Fermions** with λ = -½ have antiparallel spin and momentum
  - **Chirality** ("*Handedness*")
    - Left-handed fermions are described via: $\psi_L = P_L \psi$
    - Right-handed fermions are described via: $\psi_R = P_R \psi$

For E >> m, P_L is the projection operator for negative helicity and P_R is the projection operator for positive helicity

Dirac equation

- **Helicity and chirality:**
  - **Chirality**:
    - The projection operators are defined via:

$$P_L = \frac{\mathbb{1} - \gamma_5}{2} = P_L^\dagger, \qquad P_R = \frac{\mathbb{1} + \gamma_5}{2} = P_R^\dagger$$

with:

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma_5^\dagger$$

$$\overline{\psi}_L = (P_L\psi)^\dagger\gamma^0 = \overline{\psi}P_R$$

$$\overline{\psi}_R = (P_R\psi)^\dagger\gamma^0 = \overline{\psi}P_L$$

    - The projection operators follow:

$$P_L = P_L^2 \qquad P_R = P_R^2$$

$$P_L P_R = P_R P_L = 0$$

- The chirality operator is defined via:

$$\gamma_5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

  - It follows:

$$\gamma_5^2 = \mathbb{1}$$

$$\gamma^\mu\gamma_5 + \gamma_5\gamma^\mu = 0$$

$$\gamma_5 P_R = -P_R$$

$$\gamma_5 P_L = -P_L$$

$$\gamma^\mu P_L = P_R\gamma^\mu$$

$$\gamma^\mu P_R = P_L\gamma^\mu$$

Maxwell equation

- Using the Maxwell equations (1864) to describe the electromagnetic field:

$$\vec{\nabla} \cdot \vec{E} = \rho$$

$$\vec{\nabla} \times \vec{B} - \frac{\partial E}{\partial t} = \vec{j} \qquad (1)$$

charge density

current density

- The electric field E and magnetic field B are constrained by two further equations:

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{and} \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (2)$$

- The components of the E and B fields can be expressed by a 3-vector A and a scalar quantity φ:

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad \text{and} \quad \vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t} \qquad (3)$$

Maxwell equation

- **These four quantities (3 components from the A vector potential and the scalar φ) transform like the components of a four-vector:**

$$A^\mu \equiv \left(A^0, \vec{A}\right) = \left(\varphi, \vec{A}\right)$$

with this definition, the equations (3) can be re-written in a covariant form:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad \textbf{(4)}$$

where the components of the field-strength tensor $F_{\mu\nu}$ are the components of the electric and magnetic fields:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -B^3 & B^2 \\ -E^2 & B^3 & 0 & -B^1 \\ -E^3 & -B^2 & B^1 & 0 \end{pmatrix}$$

$F^{\mu\nu}$ can be obtained by replacing the E^i with $-E^i$

Maxwell equation

- **Using the field-strength tensor $F_{\mu\nu}$ and the potential A we can rewrite the homogeneous Maxwell equations from (2) as:**

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$$

while the inhomogeneous Maxwell equations from (1) can be expressed as:

$$\partial_\mu F^{\mu\nu} = j^\nu \qquad \textbf{(5)}$$

where j^ν is a four-vector which incorporates the sources (i.e. the charge density and the current density):

$$j^\mu \equiv \left(j^0, \vec{j}\right) = \left(\rho, \vec{j}\right)$$

Maxwell equation

- **Gauge freedom:**
  - We have to deal with a certain degree of ambiguity as the ansatz

$$A'_\mu = A_\mu + \partial_\mu \lambda$$

(where λ is any function of position and time) would satisfy equation (4).

- A change of potential that has no impact on the the field is referred to as **gauge transformation**

- **Exploit gauge freedom and set an additional constraint on potential:**

$$\partial_\mu A^\mu = 0$$

← **Lorentz condition**

i.e. for:

$$\partial^\mu \partial_\mu \lambda = \Box \lambda = -\partial^\mu A_\mu$$

Maxwell equation

- **With this gauge choice we can easily combine (4) and (5) to obtain:**

$$\Box A^{\mu} = j^{\mu}$$

- **In empty space (i.e. for $j^{\mu} = 0$) the Maxwell equation changes to:**

$$\Box A^{\mu} = 0$$

Resembles the Klein-Gordon equation of a massless particle

where A^{μ} is identified as the wave function of the photon.

Maxwell equation

- **The solutions to this equation are plane waves:**

$$A^{\mu}_{+}(x) = \frac{1}{\sqrt{V}} \varepsilon_{\mu}(k, \lambda) \exp(-ik_{\mu}x^{\mu})$$

$$A^{\mu}_{-}(x) = \frac{1}{\sqrt{V}} \varepsilon^{*}_{\mu}(k, \lambda) \exp(ik_{\mu}x^{\mu})$$

With the wave vector $k^{\mu} = p^{\mu}$ and the polarisation vector ε^{μ}:

$$\varepsilon_{\mu}(\lambda = \pm 1) = \mp \frac{1}{\sqrt{2}}(0, 1, \pm i, 0)$$

which describes two transverse polarisation states. The longitudinal polarisation state was eliminated by the Lorentz gauge condition: $\partial_{\mu} A^{\mu} = 0$

Proca equation

- **The Proca equation describes (relativistic) massive gauge bosons (W^+, W^-, Z):**

$$\left(\Box + M^2\right) W^{\nu} = 0$$

- **Solutions are plane waves:**
  - Opposite to photons, the massive gauge bosons have three polarisation states.
    - Including a longitudinal polarisation and transverse helicity ($\lambda = 0$)

$$\varepsilon^{\mu}(p, \lambda = \pm 1) = \mp \frac{1}{\sqrt{2}}(0, 1, \pm i, 0)$$

$$\varepsilon^{\mu}(p, \lambda = 0) = \frac{1}{M}(p, 0, 0, E)$$

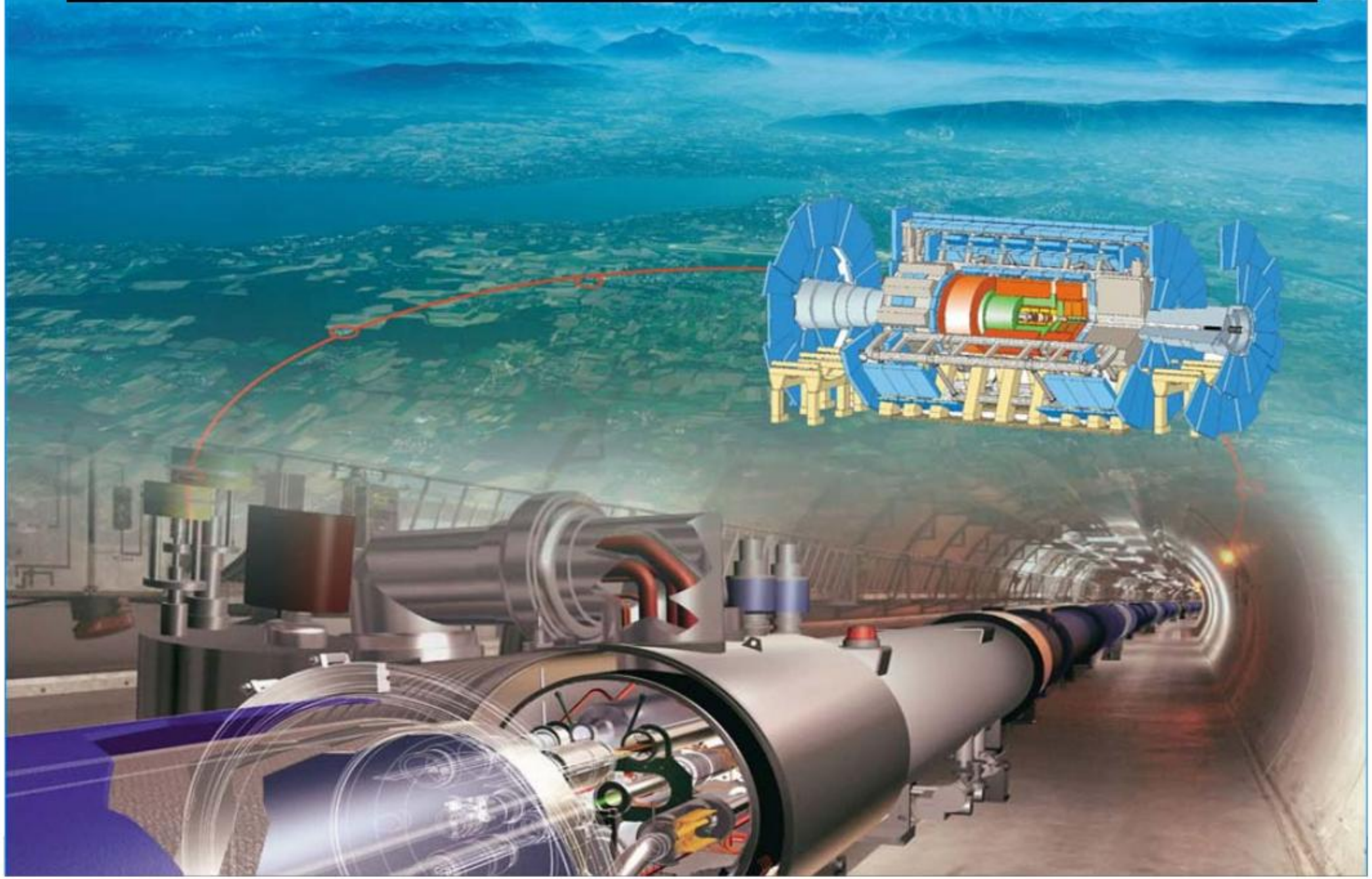
for:

$$p^{\mu} = (E, 0, 0, p)$$

The polarisation vectors are independent of the momentum for transversely polarised massive gauge bosons, but exhibit a linear dependence of the momentum for longitudinally polarised massive gauge bosons. **For high energies:**

$$\varepsilon_L^{\mu} = \frac{1}{M} p^{\mu}$$

1.2.4 Lagrange formalism



Lagrange formalism (reminder)

- **All theories of classical physics can be derived via the principle of least action:**
  - The Lagrangian and action are related by:

$$\mathcal{A} = \int_{t_1}^{t_2} dt\, L\left(q_r(t), \dot{q}_r(t), t\right)$$

    The Lagrangian L is a function of the coordinates and the velocity, while t_1 and t_2 indicate the initial and final time between which we study the system.

  - Among all trajectories that join $q(t_1)$ and $q(t_2)$, the system will follow the one for which the action is stationary:

$$\delta\mathcal{A} = 0$$

    i.e. the path for which the variation of the action vanishes.

Lagrange formalism (reminder)

- The **Euler-Lagrange equations**, i.e the equations of motions, follow from the $\delta \mathcal{A} = 0$ requirement:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) = \frac{\partial L}{\partial q_r}$$

- **Example:**
    - For a particle of mass m moving in a time-independent potential V(x), we can choose the Lagrangian as:

    $$L = \frac{1}{2} m \dot{x}^2 - V(x)$$

    The Euler-Lagrange equation as derived from the Lagrangian is:

    $$\frac{d}{dt} m \dot{x} = -\nabla V$$

    as expected from Newton’s second law.

Lagrange formalism (in field theory)

- **In field theory, the action becomes a space-time dependent integral of a Lagrangian:**

$$\mathcal{A} \equiv \int_{t_1}^{t_2} dt \underbrace{\int d^3x \mathcal{L}\left(\phi(x), \partial_\mu \phi(x)\right)}_{\text{total Lagrangian } L}$$

$$q_r(t) \rightarrow \phi(x)$$
$$\dot{q}_r(t) \rightarrow \partial_\mu \phi(x)$$

With the Lagrange density: $\mathcal{L}\left(\phi(x), \partial_\mu \phi(x)\right)$

- **Requiring the principle of least action to be fulfilled leads to the Lagrange equation:**

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} - \frac{\partial \mathcal{L}}{\partial \phi(x)} = 0$$

Lagrange formalism (examples of important Lagrange densities)

- **Scalar field (Klein-Gordon equation):**

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu \Phi)(\partial^\mu \Phi) - m^2\Phi^2\right] \quad \Rightarrow \quad \left(\Box + m^2\right)\Phi = 0$$

- **Dirac field (Dirac equation):**

$$\mathcal{L} = \bar{\Psi}\left(i\gamma^\mu \partial_\mu - m\right)\Psi \quad \Rightarrow \quad \left(i\gamma^\mu \partial_\mu - m\right)\Psi = 0$$

- **Electromagnetic field (Maxwell equation)**

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= -\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\end{aligned} \quad \Rightarrow \quad \Box A^\mu = 0$$

Lagrange formalism (examples of important Lagrange densities)

- **Massive vector bosons like W^+, W^-, Z^0 (Proca equation):**

$$\mathcal{L} = -\frac{1}{4}\left(\partial_\mu W_\nu - \partial_\nu W_\mu\right)\left(\partial^\mu W^\nu - \partial^\nu W^\mu\right) + \frac{1}{2} M^2 W^\mu W_\mu$$

$\Downarrow$

width: $\partial_\nu W^\nu = 0$

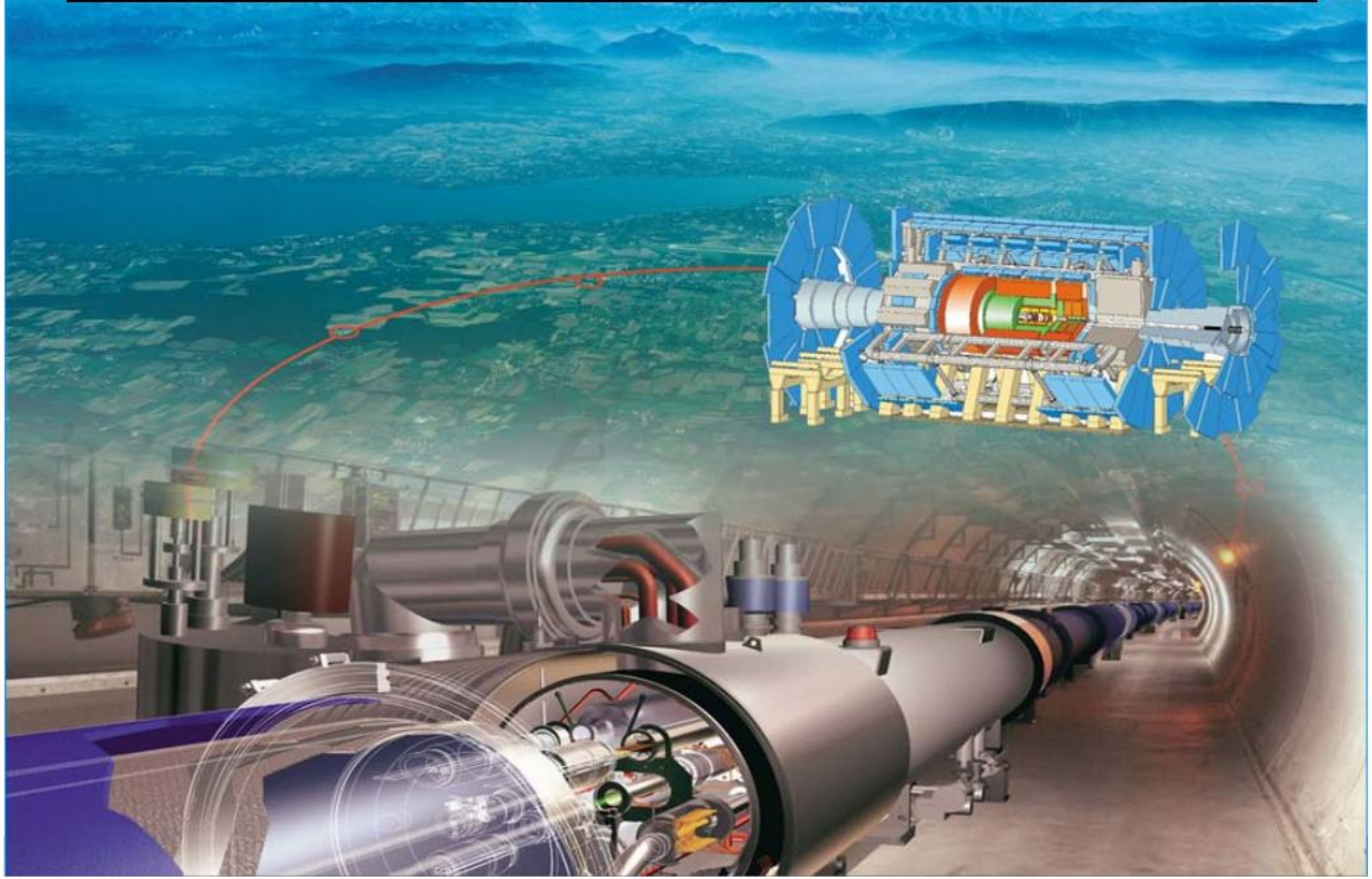
$$\left(\Box + M^2\right) W^\nu = 0$$

- **Quantum electrodynamic (QED):**

$$\mathcal{L}_{QED} = \bar{\Psi}\left(i\gamma^\mu \partial_\mu - m\right)\Psi - j^\mu A_\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

width: $j^\mu = q\bar{\Psi}\gamma^\mu\Psi$

1.2.5 Collisions



Relativistic Collisions

- **In a relativistic collision, energy and momentum are always conserved (i.e. all four components of the energy-momentum four-vector are conserved):**

    1. **Energy is conserved:** $E_A + E_B = E_C + E_D$
    2. **Momentum is conserved:** $\vec{p}_A + \vec{p}_B = \vec{p}_C + \vec{p}_D$
    3. Kinetic energy may or may not be conserved

for:

$$A + B \rightarrow C + D$$

1. and 2. can be combined into a single expression: $p^\mu_A + p^\mu_B = p^\mu_C + p^\mu_D$

- **Collisions can be classified as "sticky", "explosive" or "elastic", depending on whether the kinetic energies decreases, increases or remains the same:**
    1. **Sticky:** kinetic energy decreases, rest energy and mass increase
    2. **Explosive:** kinetic energy increases, rest energy and mass decrease
    3. **Elastic:** kinetic energy is conserved, rest energy and mass are conserved

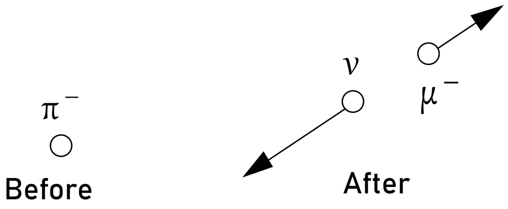
Relativistic Collisions

Note:

- Except for elastic collisions the rest mass is not conserved.
  - In the decay $\pi^0 \rightarrow \gamma + \gamma$ the initial mass was 135 MeV, but the final mass is zero. I.e. rest mass of the pion is converted into kinetic energy.
- If the rest mass of the initial particles is conserved, then a collision must have been elastic:
  - In elementary particle physics this is only the case if initial and final state particles are identical:
    - Electron-proton scattering: $e^- + p \rightarrow e^- + p$
    - Møller scattering: $e^- + e^- \rightarrow e^- + e^-$
    - Bhabha scattering: $e^- + e^+ \rightarrow e^- + e^+$

Relativistic Collisions

- **Example 1:** A pion (at rest) decays into a muon and neutrino: $\pi^+ \rightarrow \mu^+ + \nu_\mu$
  - *Question:* What is the energy of the muon ?
  - Conservation of energy and momentum require:



$$p_\pi^\lambda = p_\mu^\lambda + p_\nu^\lambda \qquad \text{or} \qquad p_\nu^\lambda = p_\pi^\lambda - p_\mu^\lambda$$

$$p_{\pi,\lambda} = p_{\mu,\lambda} + p_{\nu,\lambda} \qquad \text{or} \qquad p_{\nu,\lambda} = p_{\pi,\lambda} - p_{\mu,\lambda}$$

- Here: we use λ as the space-time index.

$$\underbrace{p_\nu^\lambda p_{\nu,\lambda}}_{=m_\nu^2} = \underbrace{p_\pi^\lambda p_{\pi,\lambda}}_{=m_\pi^2} - \underbrace{p_\mu^\lambda p_{\pi,\lambda}}_{=E_\pi E_\mu - \vec{p}_\pi \vec{p}_\mu} - \underbrace{p_\pi^\lambda p_{\mu,\lambda}}_{=E_\mu E_\pi - \vec{p}_\mu \vec{p}_\pi} + \underbrace{p_\mu^\lambda p_{\mu,\lambda}}_{=m_\mu^2}$$

$$\underbrace{m_\nu^2}_{\approx 0} = m_\pi^2 - \underbrace{E_\pi}_{m_\pi} E_\mu - \underbrace{\vec{p}_\pi}_{=0} \vec{p}_\mu - E_\mu \underbrace{E_\pi}_{=m_\pi} - \vec{p}_\mu \underbrace{\vec{p}_\pi}_{=0} + m_\mu^2$$

$$\Rightarrow \quad \boxed{E_\mu = \frac{m_\pi^2 + m_\mu^2}{2m_\pi}}$$

Relativistic Collisions

- **Example:** Production of antiprotons at the Bevatron via: $p + p \rightarrow p + p + p + \bar{p}$
  - *Question:* What is the threshold energy for this reaction ?
  - *Solution*:
    - Study left side of reaction in Lab frame:

$$p^{\mu}_{\mathrm{TOT,LAB}} = (E + m, |\vec{p}|, 0, 0)$$

    - Study right side of reaction in CM frame (with all for finale state particles being at rest):

$$p^{\mu}_{\mathrm{TOT,CM}} = (4m, 0, 0, 0)$$

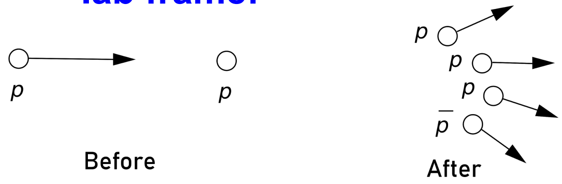
    - While both four-vectors are different, their invariant masses are not:

$$p_{\mu,\mathrm{TOT,LAB}}\, p^{\mu}_{\mathrm{TOT,LAB}} = p_{\mu,\mathrm{TOT,CM}}\, p^{\mu}_{\mathrm{TOT,CM}}$$

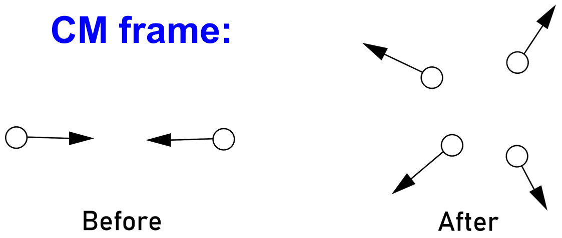
    - Thus: $(E + m)^2 - \vec{p}^2 = (4m)^2$

and finally: $E = 7m$ i.e. roughly 6 GeV

lab frame:



CM frame:



In CM-frame: $\sum_{i}^{N} \vec{p}_i = 0$