Testing the Standard Model of Elementary Particle Physics I

Third lecture

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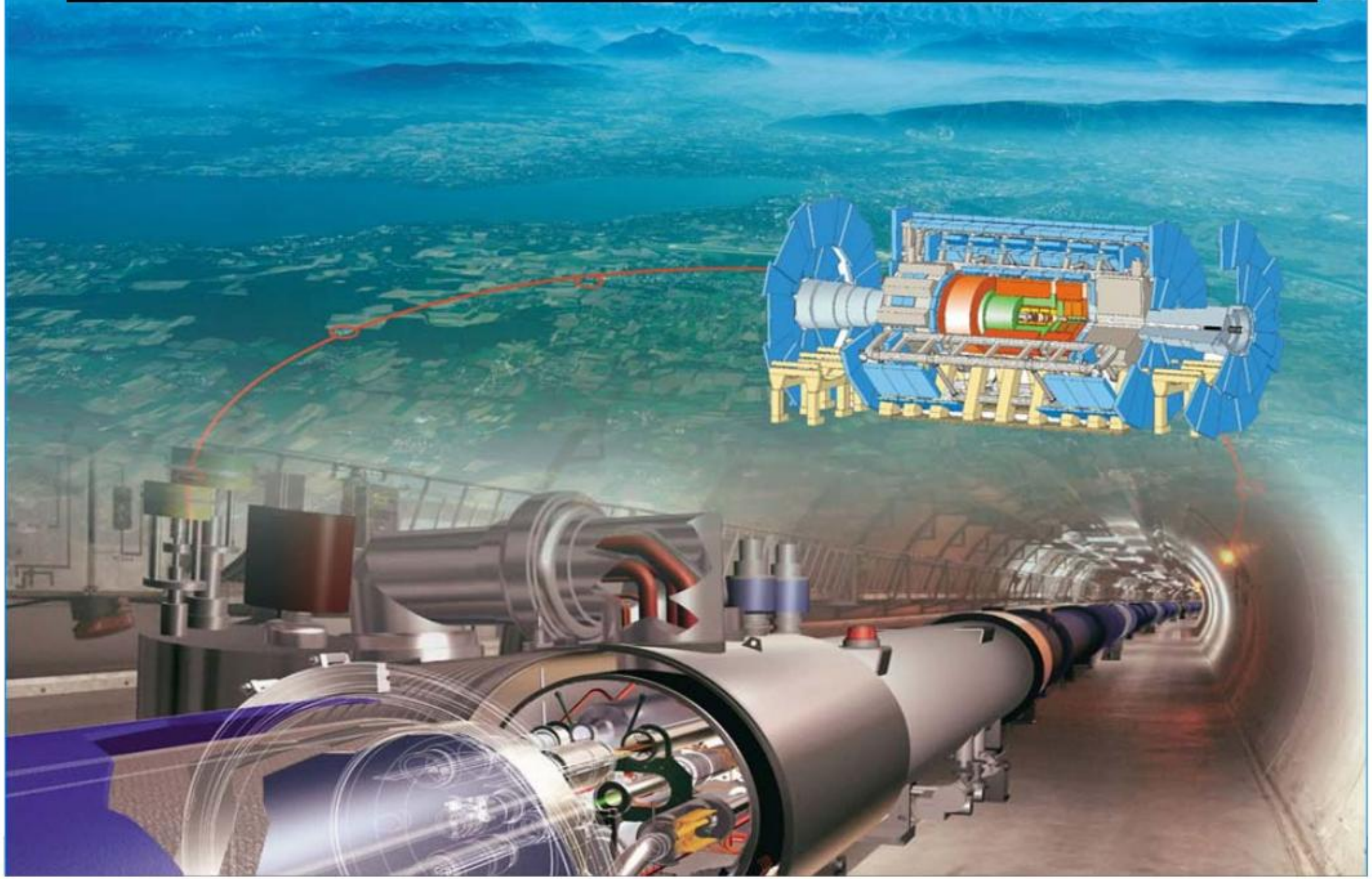
19th November 2020

1.3 Feynman Calculus



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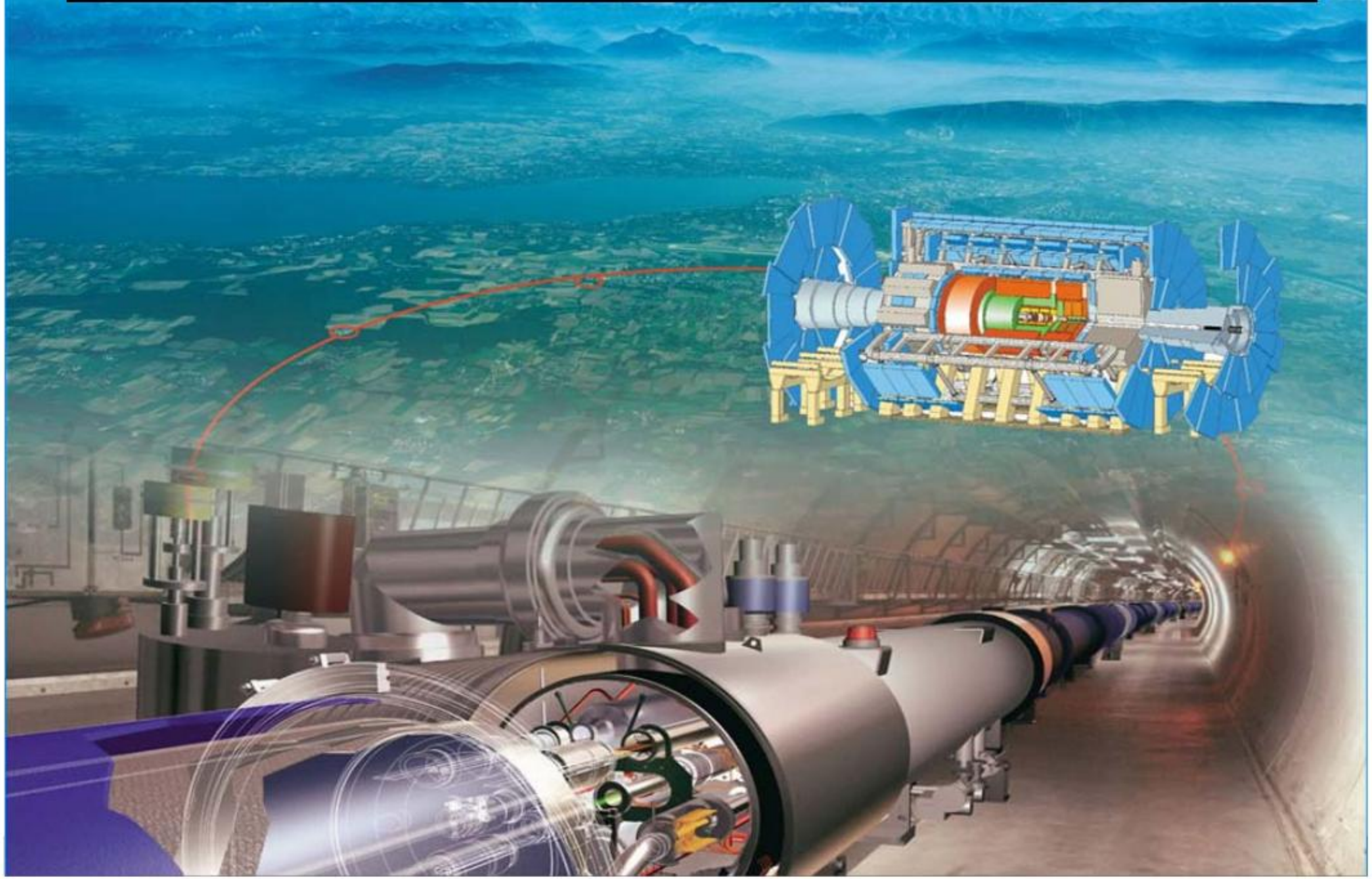
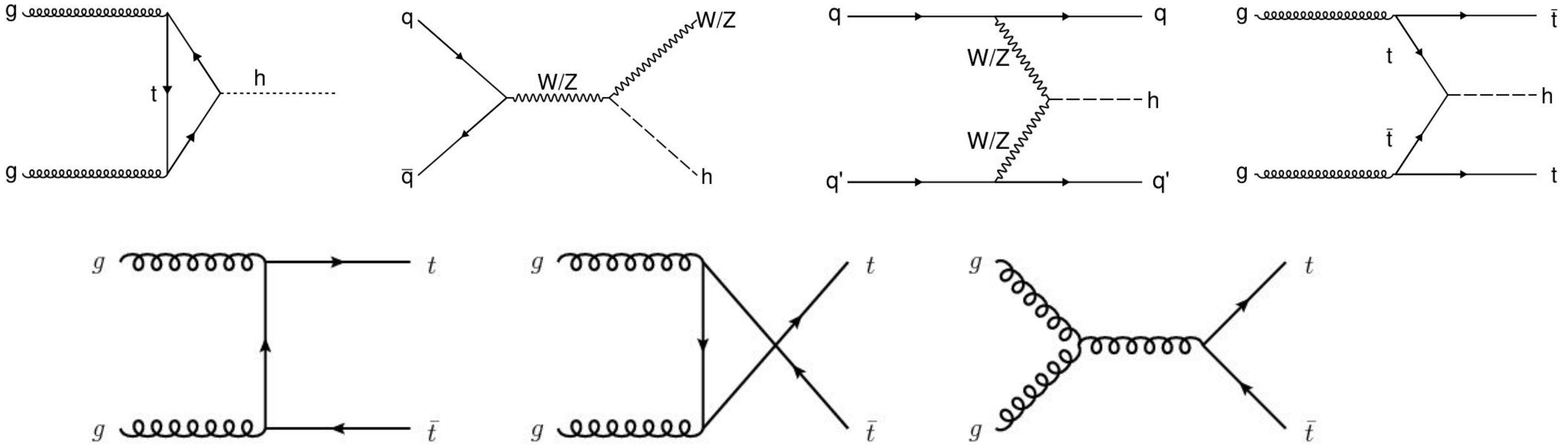
1.3.1 Introduction



Introduction:

- **Feynman-diagrams:**
  - Schematic representation of the mathematical expressions describing the behavior and interactions of elementary particles.



  - Will introduce methodology of calculating production cross section for any process based on a fixed set of rules, **Feynman rules**.

Introduction:

- **Cross sections:**
  - **The term "cross section" derives from a thought experiment involving the scattering process of hard spheres:**
    - Imagine a hard sphere of radius a, located somewhere within a total area of A
    - A second sphere is thrown towards the first sphere
      - The target sphere shows an area of πa^2
    - The probability that the incoming sphere would scatter from the target sphere is given by:

$$P_S = \frac{\pi a^2}{A}$$

Cross sectional area of the sphere in terms of the beam area

or: $\sigma = P_S A$

    - Imagine we have a parallel beam with the density ρ and the velocity v towards the target.
      - In time t, this beam will fill a volume ρvtA, where A is now the area normal to the beam which fully contains it.
        - Choosing t such that only one particle is contained in this volume, we can write:

$$1 = \rho v t A$$

Introduction:

- Thus the cross section definition can be rewritten as:

$$\sigma = \frac{P_S/t}{\rho v}$$

where P_S/t is the transition rate, i.e. the probability of scattering per unit time and ρv is the flux of particles.

⬇

A cross sections can be understood as a transition rate per unit of particle flux

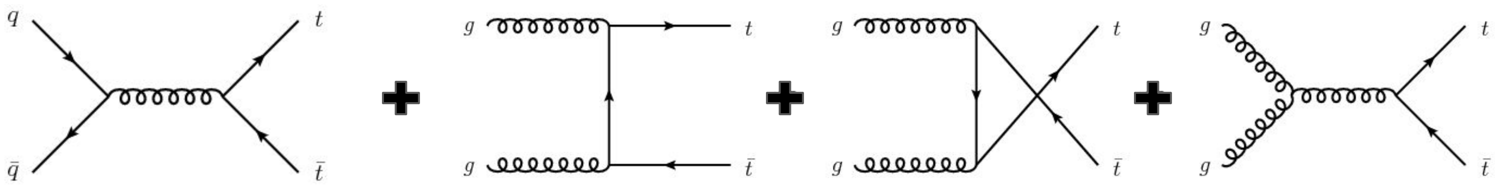
Have to migrate these expressions to quantum mechanical scattering processes

Introduction:

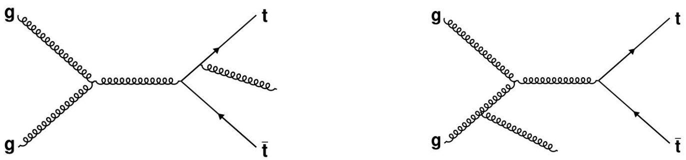
- **Cross sections:**
  - **Calculation of inclusive cross section:**

$$\sigma_{\text{tot}} = \sum_{i}^{N} \sigma_i$$

    - Contributions from various production modes.

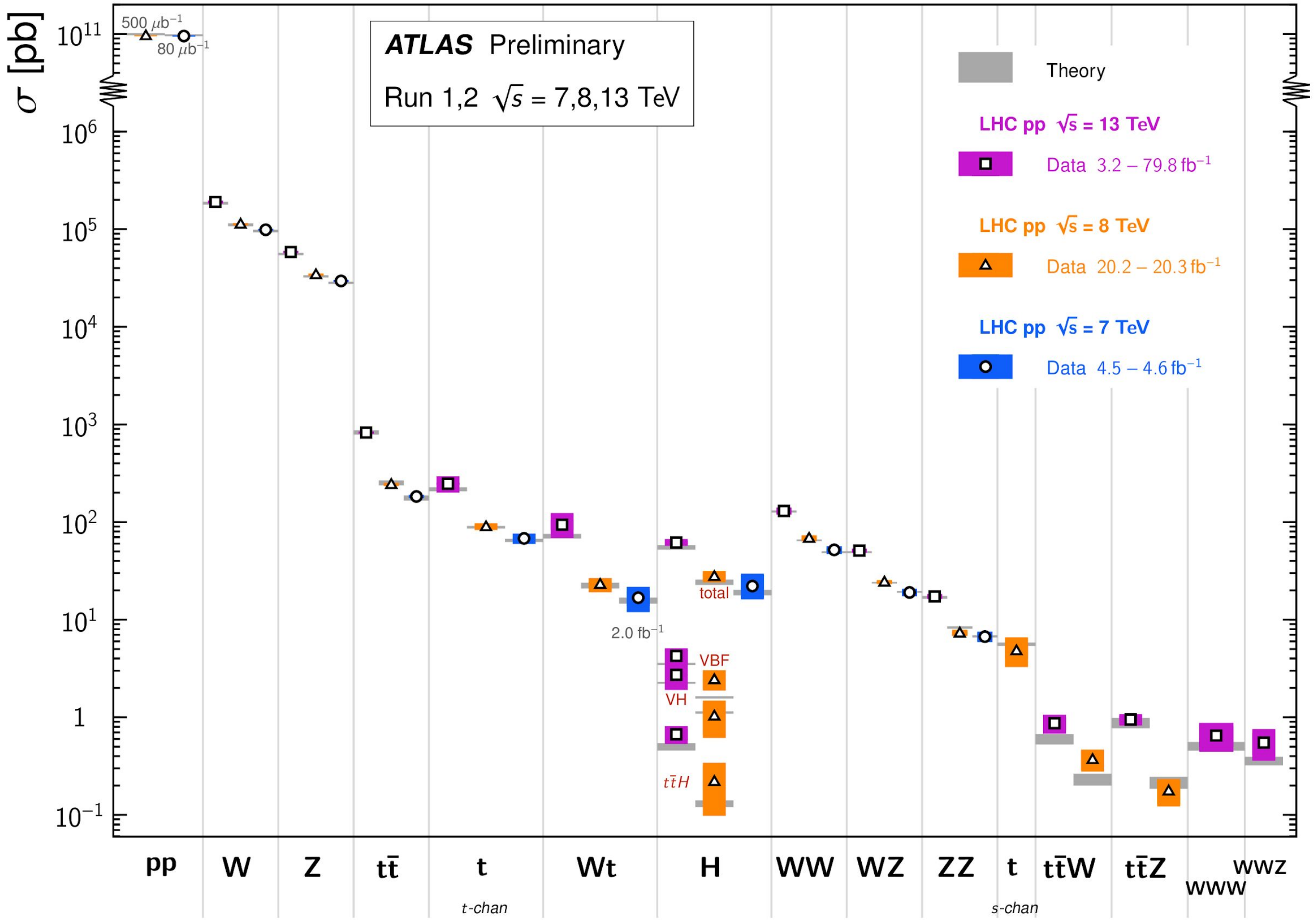


    - Tree level diagrams and ideally higher order diagrams:

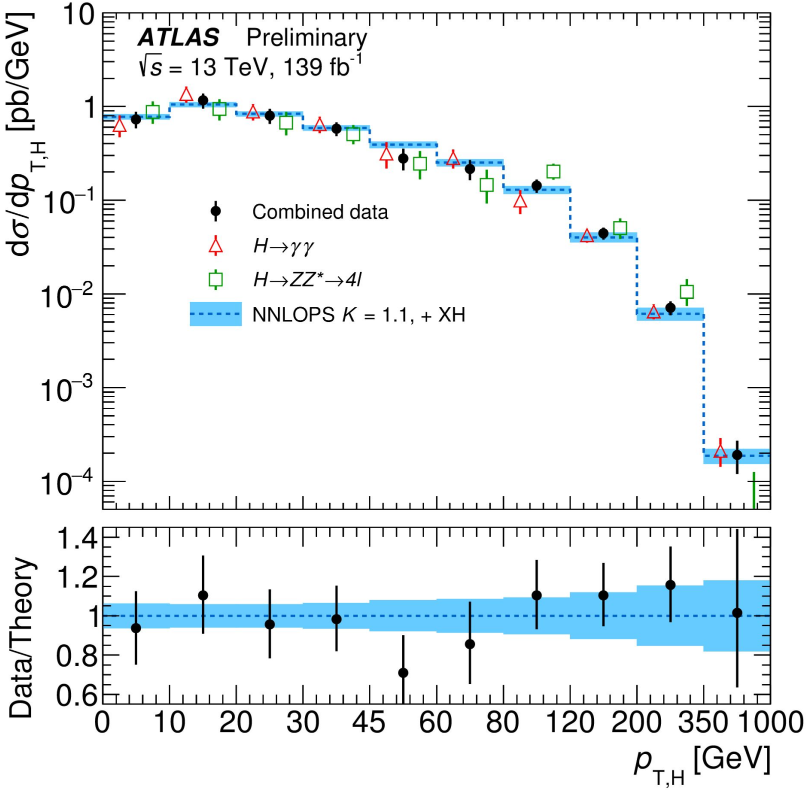


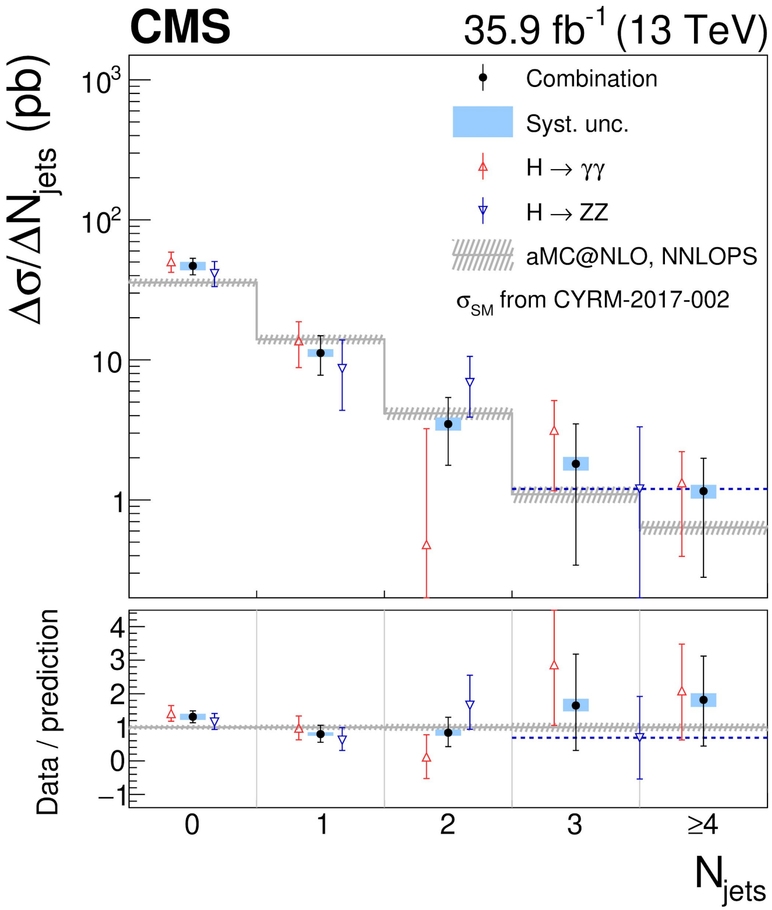
Standard Model Total Production Cross Section Measurements

Status: May 2020



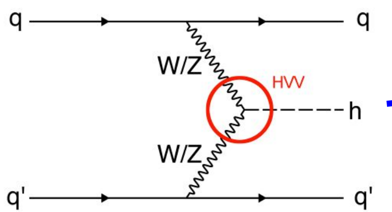
Differential cross sections:

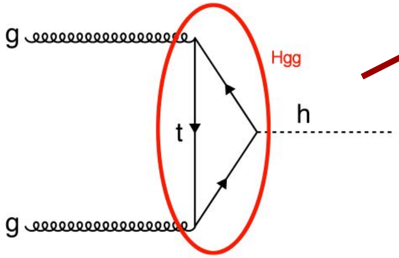


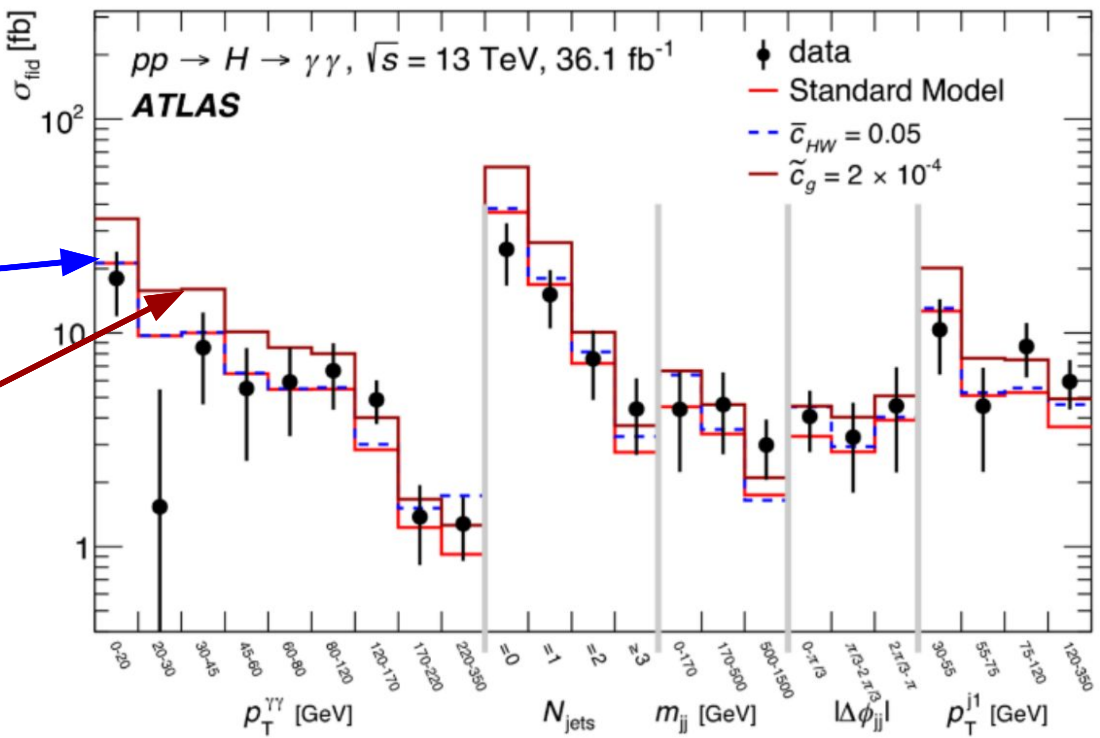


Differential cross sections as a gate to new physics

BSM physics contributions to the Higgs boson production process can modulate certain kinematical observables







Perform hypothesis tests to determine whether data fits better to SM predictions or BSM hypotheses

Introduction:

- **Decay rates and branching fraction:**
  - The **decay rate** is defined as the probability per unit time that a decay $X \rightarrow x_1x_2$ will occur
    - Decay rate and mean lifetime of a particle are related via:

$$\tau = \frac{1}{\Gamma}$$

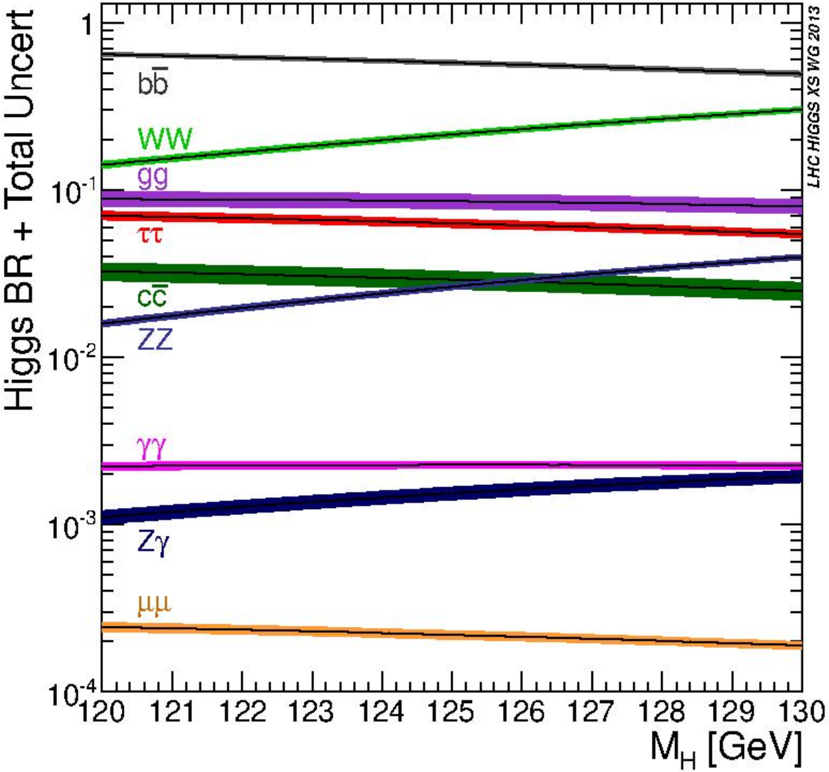
    - For particles with multiple decay modes, the total decay rate is the sum of the individual decay rates:

$$\Gamma_{\text{tot}} = \sum_i^N \Gamma_i \quad \Rightarrow \quad \tau = \frac{1}{\Gamma_{\text{tot}}}$$

  - The **branching ratio** of a decay $X \rightarrow x_1x_2$ is defined via:

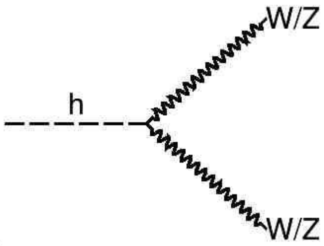
$$BR(X \rightarrow x_1x_2) = \frac{\Gamma_i}{\Gamma_{\text{tot}}}$$

Higgs boson decays



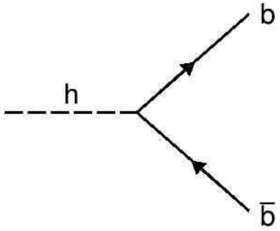
Decay mode	Branching fraction [%]
$H \to b\bar{b}$	57.5 ± 1.9
$H \to WW^*$	21.6 ± 0.9
$H \to gg$	8.56 ± 0.86
$H \to \tau\tau$	6.30 ± 0.36
$H \to c\bar{c}$	2.90 ± 0.35
$H \to ZZ^*$	2.67 ± 0.11
$H \to \gamma\gamma$	0.228 ± 0.011
$H \to Z\gamma$	0.155 ± 0.014
$H \to \mu\mu$	0.022 ± 0.001

Higgs boson decays

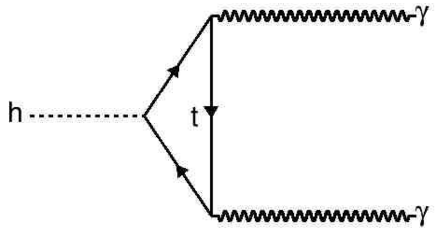


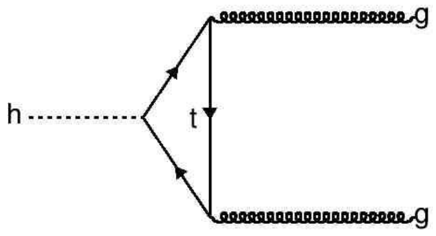
- Strength of the coupling between the Higgs boson and other particles is proportional to the particle mass:

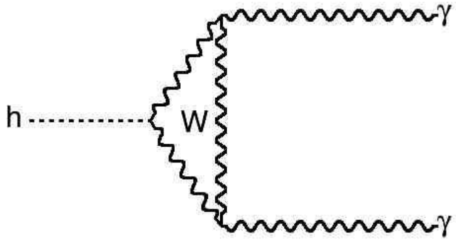
$$\mathcal{L}_{Hff} = -\frac{m_f}{v} h f \bar{f} \qquad \text{and} \qquad \mathcal{L}_{HVV} = \frac{1}{v}\left(2m_W^2 W_\mu^+ W^{-\mu} + 2m_Z^2 Z_\mu Z^\mu\right) h$$

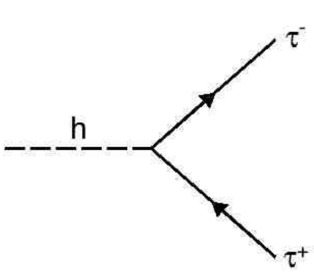


- Thus decays to massless particles such as photon or gluons is only possible via top quark (or W boson) loops
- The masses of the particles running in these loops are large and thus such decay modes can compete with decays to fermions or W and Z bosons









W/Z boson decays

- **Lepton universality:**
  - All three types of charged leptons interact in the same way with other particles.
  - The three lepton types are created equally often in particle transformations, or decays (once differences in their mass are accounted for)

Decay Mode	BR
$Z \to e^+e^-$	$(3.3632 \pm 0.0042)\%$
$Z \to \mu^+\mu^-$	$(3.3662 \pm 0.0066)\%$
$Z \to \tau^+\tau^-$	$(3.3696 \pm 0.0083)\%$
$Z \to$ invisible	$(20.000 \pm 0.055)\%$
$Z \to$ hadrons	$(69.911 \pm 0.056)\%$

Decay Mode	BR
$W \to e\nu$	$(10.71 \pm 0.16)\%$
$W \to \mu\nu$	$(10.63 \pm 0.15)\%$
$W \to \tau\nu$	$(11.38 \pm 0.21)\%$
$W \to$ hadrons	$(67.41 \pm 0.27)\%$

Quark decays

- **CKM matrix elements describe transition from one quark flavour to another:**
  - I.e. V_{ij} measures the coupling of quark i to quark j:
  - The CKM matrix is given via:

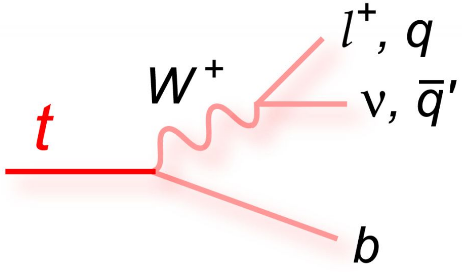
$$V_{\mathrm{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{tc} & V_{tb} \end{pmatrix}$$

  - The magnitudes of the matrix elements are:

$$\begin{pmatrix} 0.97446 \pm 0.00010 & 0.22452 \pm 0.00044 & 0.00365 \pm 0.00012 \\ 0.22438 \pm 0.00044 & 0.97359^{+0.00010}_{-0.00011} & 0.04214 \pm 0.00076 \\ 0.00896^{+0.00024}_{-0.00023} & 0.04133 \pm 0.00074 & 0.999105 \pm 0.000032 \end{pmatrix}$$

- Top quark decays almost exclusively via $t \rightarrow bW$

$$|V_{tb}| \approx 1$$



Latest measurements of CKM matrix elements taken from: https://pdg.lbl.gov/2019/reviews/rpp2019-rev-ckm-matrix.pdf

Lepton decays

Decay Mode	BR
$\tau^- \to e^- \nu_e \nu_\tau$	$(17.83 \pm 0.04)\%$
$\tau^- \to \mu^- \nu_\mu \nu_\tau$	$(17.41 \pm 0.04)\%$
$\tau^- \to \pi^- \pi^0 \nu_\tau$	$(25.52 \pm 0.09)\%$
$\tau^- \to \pi^- \nu_\tau$	$(10.83 \pm 0.06)\%$
$\tau^- \to \pi^- \pi^0 \pi^0 \nu_\tau$	$(9.30 \pm 0.11)\%$
$\tau^- \to \pi^- \pi^+ \pi^- \nu_\tau$	$(8.99 \pm 0.05)\%$
$\tau^- \to \pi^- \pi^+ \pi^- \pi^0 \nu_\tau$	$(2.74 \pm 0.07)\%$
$\tau^- \to \pi^- \pi^0 \pi^0 \pi^0 \nu_\tau$	$(1.04 \pm 0.07)\%$

Decay Mode	BR
$\mu^- \to e^- \bar{\nu}_e \nu_\mu$	100%

- **Electrons are stable**
- **Lifetimes:**
  - Muons: $2.2 \cdot 10^{-6}$ s
  - Taus: $290.6 \cdot 10^{-15}$ s
- **Neutrino oscillate:**
  - Will be discussed next semester

The Golden rule for scattering and decays

- **There are two ingredients to the calculation of cross sections and decay rates:**
  1. **The Amplitude for the process:** $\mathcal{M}$
  2. **The available phase space**
- **The amplitude contains all the dynamical information of the process**
  - It will be calculated by evaluating all relevant Feynman diagrams using a fixed set of rules (i.e. the “Feynman rules”)
- **The phase space is purely kinematical**
  - It depends on masses, energies and momenta of particles participating in a reaction
  - Reflects the fact that a given process is more likely to occur the more phase space is available:
    - A decay of a heavy particle into light secondaries involves a large phase space factor, as there are many different way to apportion the available energies.
    - The neutron decay via $n \to p + e + \bar{\nu}_e$ is highly suppressed as there is almost no mass to spare and thus the phase space factor is vers small

1.3.2 Golden Rule for scattering

Golden Rule for scattering

- **The production cross section σ for the scattering of two particles with given 4-momenta p_1 and p_2 which produces several particles in the final state**

$$X_1 + X_2 \to X_3 + X_4 + ... + X_n$$

is given via:

$$\sigma = \frac{S}{4\sqrt{(p_1 p_2)^2 - (m_1 m_2)^2}} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4 \left(p_1 + p_2 - p_3 - p_4 - \ldots - p_n\right) \times \prod_{j=3}^{n} 2\pi\delta\left(p_j^2 - m_j^2\right) \theta(p_j^0) \frac{d^4 p_j}{(2\pi)^4} \quad \textbf{(6)}$$

The integral is over the outgoing particle momenta

Golden Rule for scattering

- **The production cross section σ for the scattering of two particles with given 4-momenta p_1 and p_2 which produces several particles in the final state**

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First delta function ensures energy and momentum conservation between the initial and final state.

The integral is over the outgoing particle momenta

Golden Rule for scattering

- **The production cross section σ for the scattering of two particles with given 4-momenta p_1 and p_2 which produces several particles in the final state**

$$X_1 + X_2 \to X_3 + X_4 + \dots + X_n$$

is given via:

$$\sigma = \frac{S}{4\sqrt{(p_1 p_2)^2 - (m_1 m_2)^2}} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4 \left(p_1 + p_2 - p_3 - p_4 - \dots - p_n\right)$$
$$\times \prod_{j=3}^{n} 2\pi\delta\left(p_j^2 - m_j^2\right) \theta(p_j^0) \frac{d^4 p_j}{(2\pi)^4} \qquad \textbf{(6)}$$

Second delta function ensures that the outgoing particles are real, i.e., on their mass-shell.

The integral is over the outgoing particle momenta

Golden Rule for scattering

- **The production cross section σ for the scattering of two particles with given 4-momenta p_1 and p_2 which produces several particles in the final state**

$$X_1 + X_2 \to X_3 + X_4 + ... + X_n$$

is given via:

$$\sigma = \frac{S}{4\sqrt{(p_1 p_2)^2 - (m_1 m_2)^2}} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4 \left(p_1 + p_2 - p_3 - p_4 - \ldots - p_n\right)$$

$$\times \prod_{j=3}^{n} 2\pi\delta\left(p_j^2 - m_j^2\right) \theta(p_j^0) \frac{d^4 p_j}{(2\pi)^4} \qquad \textbf{(6)}$$

The theta function leads to positive outgoing particles energies.

The integral is over the outgoing particle momenta

Golden Rule for scattering

- **The production cross section σ for the scattering of two particles with given 4-momenta p_1 and p_2 which produces several particles in the final state**

$$X_1 + X_2 \to X_3 + X_4 + ... + X_n$$

is given via:

$$\sigma = \frac{S}{4\sqrt{(p_1 p_2)^2 - (m_1 m_2)^2}} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4 \left(p_1 + p_2 - p_3 - p_4 - ... - p_n\right)$$

$$\times \prod_{j=3}^{n} 2\pi\delta\left(p_j^2 - m_j^2\right) \theta(p_j^0) \frac{d^4 p_j}{(2\pi)^4} \qquad \textbf{(6)}$$

The dynamics of the scattering process are described via the scattering amplitude

The integral is over the outgoing particle momenta

Golden Rule for scattering

- **The production cross section σ for the scattering of two particles with given 4-momenta p_1 and p_2 which produces several particles in the final state**

$$X_1 + X_2 \to X_3 + X_4 + ... + X_n$$

is given via:

$$\sigma = \frac{S}{4\sqrt{(p_1 p_2)^2 - (m_1 m_2)^2}} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4 \left(p_1 + p_2 - p_3 - p_4 - \ldots - p_n\right)$$

$$\times \prod_{j=3}^{n} 2\pi\delta\left(p_j^2 - m_j^2\right)\theta(p_j^0)\frac{d^4 p_j}{(2\pi)^4} \quad \textbf{(6)}$$

S is a statistical factor which accounts for identical particles in the final state. For each group g_i of identical final state particles, S contains a factor of $\frac{1}{g_i!}$. Thus, if a + b → c + c + d + d + d , then S = (1/2!)(1/3!) = 1/12

The integral is over the outgoing particle momenta

Golden Rule for scattering

- **Equation (6) can be brought into a more suitable form by re-writing the second delta functions:**

$$\delta\left(p^2 - m^2\right) = \delta\left(E^2 - \vec{p}^2 - m^2\right) = \delta(E^2 - (\vec{p}^2 + m^2)) \quad \textbf{(7)}$$

- **One can exploit the following property of delta functions:**

$$\delta\left(x^2 - a^2\right) = \frac{1}{2a}\left[\delta(x - a) + \delta\left(x + a\right)\right]$$

with $x = E = p^0$ **and** $a = \sqrt{\vec{p}^2 + m^2}$ **one can rewrite equation (7) as:**

$$\delta(p^2 - m^2) = \frac{1}{2\sqrt{\vec{p}^2 + m^2}}\left[\delta\left(E - \sqrt{\vec{p}^2 + m^2}\right) + \underbrace{\delta\left(E + \sqrt{\vec{p}^2 + m^2}\right)}_{*}\right]$$

Where (*) does not contribute to the integral in (6) (θ function ensures E > 0)

Golden Rule for scattering

- **Thus one obtains:**

$$\sigma = \frac{S}{4\sqrt{(p_1 p_2)^2 - (m_1 m_2)^2}} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4 \left(p_1 + p_2 - p_3 - p_4 - \ldots - p_n\right)$$

$$\times \prod_{j=3}^{n} 2\pi \frac{1}{2\sqrt{\vec{p}_j^2 + m_j^2}} \delta\left(E_j + \sqrt{\vec{p}_j^2 + m_j^2}\right) \theta(p_j^0) \frac{d^4 p_j}{(2\pi)^4}$$

- **Perform integration over $p_j^0 = E_j$ such that**

$$\sigma = \frac{S}{4\sqrt{(p_1 p_2)^2 - (m_1 m_2)^2}} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4 \left(p_1 + p_2 - p_3 - p_4 - \ldots - p_n\right)$$

$$\times \prod_{j=3}^{n} \frac{1}{2\sqrt{\vec{p}_j^2 + m_j^2}} \frac{d^3 \vec{p}_j}{(2\pi)^3} \qquad \textbf{(8)}$$

follows.

with: $E_j = \sqrt{\vec{p}_j^2 + m_j^2}$

Golden Rule for scattering

- **Example (2 → 2 scattering):** $X_1 + X_2 \rightarrow X_3 + X_4$
  - First calculate $\sqrt{(p_1 p_2)^2 - (m_1 m_2)^2}$, which is a Lorentz-invariant scalar that can be evaluated in any coordinate system. The centre-of-mass system, is for this task particularly convenient: $\vec{p}_2^{\,*} = -\vec{p}_1^{\,*}$
    - Such that:

$$p_1 \cdot p_2 = E_1^* E_2^* + \vec{p}^{\,*2}$$

$$m_i^2 = E_i^{*2} - \vec{p}^{\,*2}$$

$$m_1^2 \cdot m_2^2 = E_1^{*2} E_2^{*2} - E_1^{*2} \vec{p}^{\,*2} - \vec{p}^{\,*2} E_2^{*2} + \vec{p}^{\,*4}$$

$$(p_1 \cdot p_2)^2 - m_1^2 \cdot m_2^2 = E_1^* \vec{p}^{\,*2} + \vec{p}^{\,*2} E_2^{*2} + 2 E_1^* E_2^* \vec{p}^{\,*2}$$

$$= \vec{p}^{\,*2} (E_1^* + E_2^*)^2$$

Golden Rule for scattering

- **Example (2 → 2 scattering):** $X_1 + X_2 \rightarrow X_3 + X_4$
  - With these expressions equation **(8)** can be rewritten as:

$$\sigma = \frac{S}{64\pi^2 \left(E_1^* + E_2^*\right) |\vec{p}^{\,*}|} \int |\mathcal{M}|^2 \delta^4 \left(p_1^* + p_2^* - p_3^* - p_4^*\right) \frac{d^3 \vec{p}_3^{\,*}}{\sqrt{\vec{p}_3^{\,*2} + m_3^2}} \frac{d^3 \vec{p}_4^{\,*}}{\sqrt{\vec{p}_4^{\,*2} + m_4^2}} \qquad \textbf{(9)}$$

  - The four-dimensional delta function separates into an energy part and a momentum part:

$$\delta^4 \left(p_1^* + p_2^* - p_3^* - p_4^*\right) = \delta \left(E_1^* + E_2^* - E_3^* - E_4^*\right) \underbrace{\delta^3 \left(0 - \vec{p}_3^{\,*} - \vec{p}_4^{\,*}\right)}_{\delta^3 \left(\vec{p}_3^{\,*} + \vec{p}_4^{\,*}\right)}$$

due to $\vec{p}_4^{\,*} = -\vec{p}_3^{\,*}$ equation **(9)** can be written as:

$$\sigma = \frac{S}{(8\pi)^2 \left(E_1^* + E_2^*\right) |\vec{p}^{\,*}|} \int |\mathcal{M}|^2 \frac{\delta \left(E_1^* + E_2^* - \sqrt{\vec{p}_3^{\,*2} + m_3^2} - \sqrt{\vec{p}_3^{\,*2} + m_4^2}\right)}{\sqrt{\vec{p}_3^{\,*2} + m_3^2} \sqrt{\vec{p}_3^{\,*2} + m_4^2}} d^3 \vec{p}_3^{\,*}$$

Golden Rule for scattering

- **Example (2 → 2 scattering):** $X_1 + X_2 \rightarrow X_3 + X_4$
  - Introduce spherical coordinates to solve the integral:

$$\vec{p}_3 = |\vec{p}_3^{\,*}| \cdot \begin{pmatrix} \sin\theta^* \cos\phi^* \\ \sin\theta^* \sin\phi^* \\ \cos\theta^* \end{pmatrix}$$

and: $$\frac{d(\cos\theta^*)}{d\theta^*} = -\sin\theta^*$$

$$r \equiv |\vec{p}_3^{\,*}|$$

$$d\Omega^* = d\phi^* d(\cos\theta^*)$$

  - such that:

$$d^3\vec{p}_3^{\,*} = r^2 dr \; d\Omega^*$$

  - With $\sigma = \int d\Omega^* \frac{d\sigma}{d\Omega^*}$, the differential cross section is obtained as:

$$\frac{d\sigma}{d\Omega^*} = \frac{S}{(8\pi)^2 (E_1^* + E_2^*) |\vec{p}^{\,*}|} \int |\mathcal{M}|^2 \frac{\delta\left(E_1^* + E_2^* - \sqrt{r^2 + m_3^2} - \sqrt{r^2 + m_4^2}\right)}{\sqrt{r^2 + m_3^2}\sqrt{r^2 + m_4^2}} r^2 dr \quad \textbf{(10)}$$

Golden Rule for scattering

- **Example (2 → 2 scattering):** $X_1 + X_2 \to X_3 + X_4$
  - Change from r to the variable u:

$$u \equiv \sqrt{r^2 + m_3^2} + \sqrt{r^2 + m_4^2}$$

$$\frac{du}{dr} = \frac{r}{\sqrt{r^2 + m_3^2}} + \frac{r}{\sqrt{r^2 + m_4^2}} = \frac{ru}{\sqrt{r^2 + m_3^2}\sqrt{r^2 + m_4^2}} \qquad \textbf{(11)}$$

  - With this the integral from equation (10) can be rewritten as:

$$\int |\mathcal{M}|^2 \delta\left(E_1^* + E_2^* - u\right) \frac{r}{u} du$$

Upon integration, the delta-function sends u to the centre-of-mass energy of the collision:

$$u = E_1^* + E_2^* = E_{\mathrm{CM}}$$

  - From **(11)** it follows then after some algebra that:

Final state momentum

$$r = \frac{1}{2E_{\mathrm{CM}}} \sqrt{E_{\mathrm{CM}}^4 + m_3^4 + m_4^4 - 2m_3^2 m_4^2 - 2E_{\mathrm{CM}}^2\left(m_3^2 + m_4^2\right)} = |\vec{p}_3^{\,*}| \equiv |\vec{p}_f^{\,*}|$$

Golden Rule for scattering

- **Example (2 → 2 scattering):** $X_1 + X_2 \to X_3 + X_4$
  - In summary, the cross section for a 2 → 2 scattering process is given by

$$\frac{d\sigma}{d\Omega^*} = \frac{S}{(8\pi)^2 E_{CM}^2} \frac{|\vec{p}_f^{\,*}|}{|\vec{p}_i^{\,*}|} |\mathcal{M}|^2$$

  - For elastic scattering, the expression simplifies to:

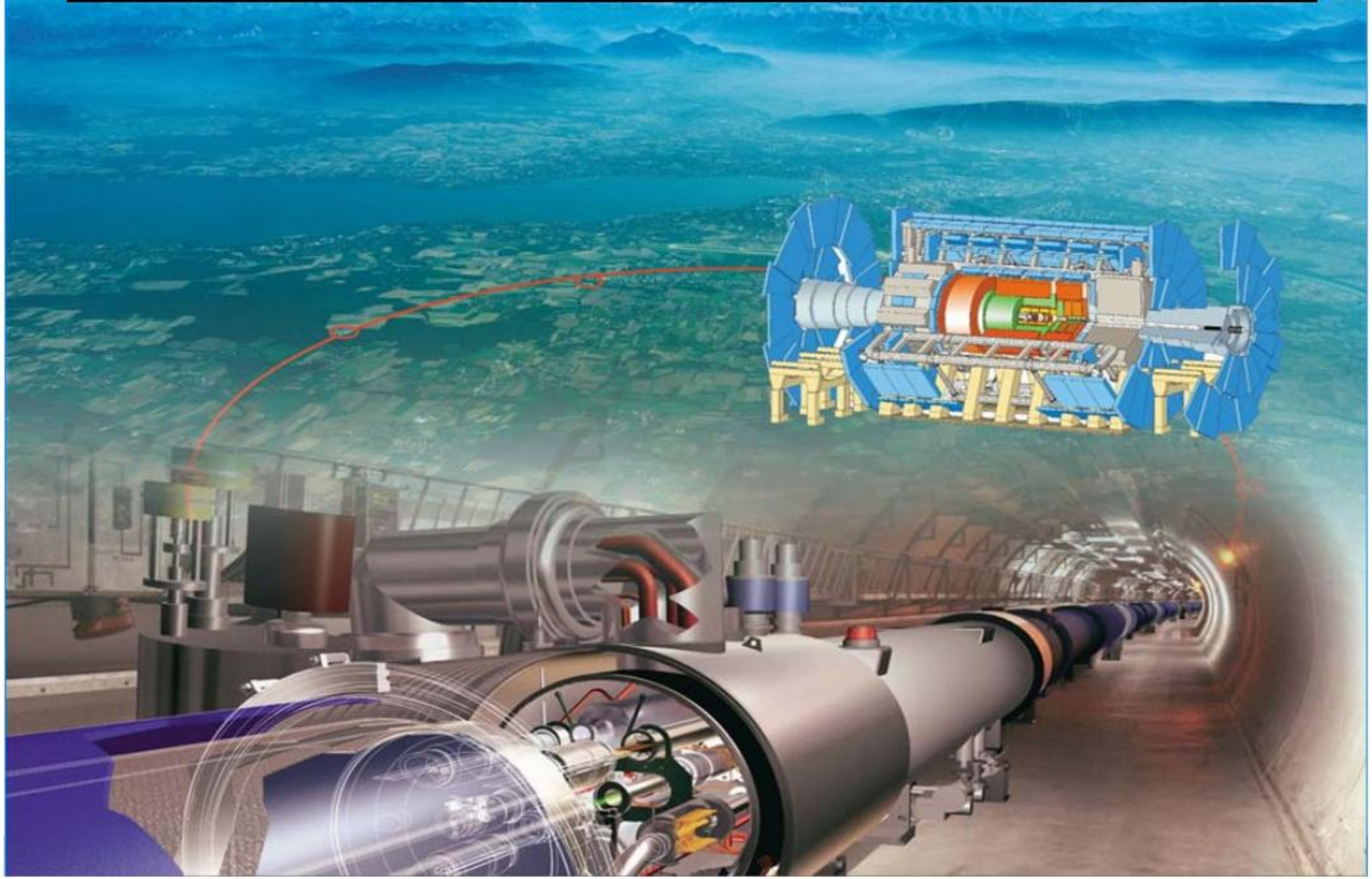
$$\frac{d\sigma}{d\Omega^*} = \frac{S}{(8\pi)^2 E_{CM}^2} |\mathcal{M}|^2$$

as: $|\vec{p}_f^{\,*}| = |\vec{p}_i^{\,*}|$

  - If there are no identical particles in the final state, the permutation factor is S = 1
  - Energy-momentum conservation implies that the only free parameters are the two angles θ^* and ϕ which specify the flight direction of particles 3.
    - Thus the cross section depends on these angles:

$$\frac{d\sigma}{d\Omega^*} = \frac{d^2\sigma}{\Delta\phi^* d(\cos\theta^*)}$$

1.3.3 Golden Rule for decays



Golden Rule for decays

- **The decay rate of a particle x_1 (at rest) with a four-momentum p_1 that decays via:**

$$X_1 \rightarrow X_2 + X_3 + ... + X_n$$

is given by the formula:

$$\Gamma = \frac{S}{2m_1} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4 \left(p_1 - p_2 - p_3 - ... - p_n\right) \times \prod_{j=2}^{n} 2\pi\delta\left(p_j^2 - m_j^2\right) \theta(p_j^0) \frac{d^4 p_j}{(2\pi)^4}$$

- **Using the same approaches as for scattering, one obtains:**

$$\Gamma = \frac{S}{2m_1} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4 \left(p_1 - p_2 - p_3 - ... - p_n\right) \times \prod_{j=2}^{n} \frac{1}{2\sqrt{\vec{p}_j^2 + m_j^2}} \frac{d^3 \vec{p}_j}{(2\pi)^3}$$

Golden Rule for decays

- **Example (Two-particle decay):** $X_1 \to X_2 + X_3$
  - If there are only two particles in the final state:

$$\Gamma = \frac{S}{32\pi^2 m_1} \int |\mathcal{M}|^2 \frac{\delta^4 \left(p_1 - p_2 - p_3\right)}{\sqrt{\vec{p}_2^{\,2} + m_2^2}\sqrt{\vec{p}_3^{\,2} + m_3^2}} d^3\vec{p}_2 \times d^3\vec{p}_3 \qquad \textbf{(12)}$$

  - Separate again the four-dimensional delta function into an energy part and a momentum part:

$$\delta^4 \left(p_1 - p_2 - p_3\right) = \delta \left(E_1 - E_2 - E_3\right) \delta^3 \left(\vec{p}_1 - \vec{p}_2 - \vec{p}_3\right)$$

  - With:

$$\vec{p}_1 = 0 \qquad \textbf{and} \qquad E_1 = m_1 \qquad \textbf{and} \qquad \delta(-x) = \delta(x)$$

equation **(12)** can be written as:

$$\Gamma = \frac{S}{32\pi^2 m_1} \int |\mathcal{M}|^2 \frac{\delta \left(m_1 - \sqrt{\vec{p}_2^{\,2} + m_2^2} - \sqrt{\vec{p}_3^{\,2} + m_3^2}\right)}{\sqrt{\vec{p}_2^{\,2} + m_2^2}\sqrt{\vec{p}_3^{\,2} + m_3^2}} \delta^3 \left(\vec{p}_2 + \vec{p}_3\right) d^3\vec{p}_2 d^3\vec{p}_3$$

Golden Rule for decays

- **Example (Two-particle decay):** $X_1 \to X_2 + X_3$
  - The $\vec{p}_3$ integral is now trivial: in view of the final delta function it simply makes the replacement:

$$\vec{p}_3 \to -\vec{p}_2$$

    which leads to:

$$\Gamma = \frac{S}{32\pi^2 m_1} \int |\mathcal{M}|^2 \frac{\delta\left(m_1 - \sqrt{\vec{p}_2^{\,2} + m_2^2} - \sqrt{\vec{p}_2^{\,2} + m_3^2}\right)}{\sqrt{\vec{p}_2^{\,2} + m_2^2}\sqrt{\vec{p}_2^{\,2} + m_3^2}} d^3\vec{p}_2$$

  - Switch again to spherical coordinates (and perform integral over angles):

with: $r \equiv |\vec{p}_2|$

$$\Gamma = \frac{S}{8\pi m_1} \int |\mathcal{M}|^2 \frac{\delta\left(m_1 - \sqrt{r^2 + m_2^2} - \sqrt{r^2 + m_3^2}\right)}{\sqrt{r^2 + m_2^2}\sqrt{r^2 + m_3^2}} r^2 dr$$

  - Change from r to the variable u:

$$u \equiv \sqrt{r^2 + m_2^2} + \sqrt{r^2 + m_3^2}$$

Golden Rule for decays

- **Example (Two-particle decay):** $X_1 \to X_2 + X_3$

  and also:

$$\frac{du}{dr} = \frac{ru}{\sqrt{r^2 + m_2^2}\sqrt{r^2 + m_3^2}}$$

  - Thus:

$$\Gamma = \frac{S}{8\pi m_1} \int |\mathcal{M}|^2 \delta(m_1 - u) \frac{r}{u} du$$

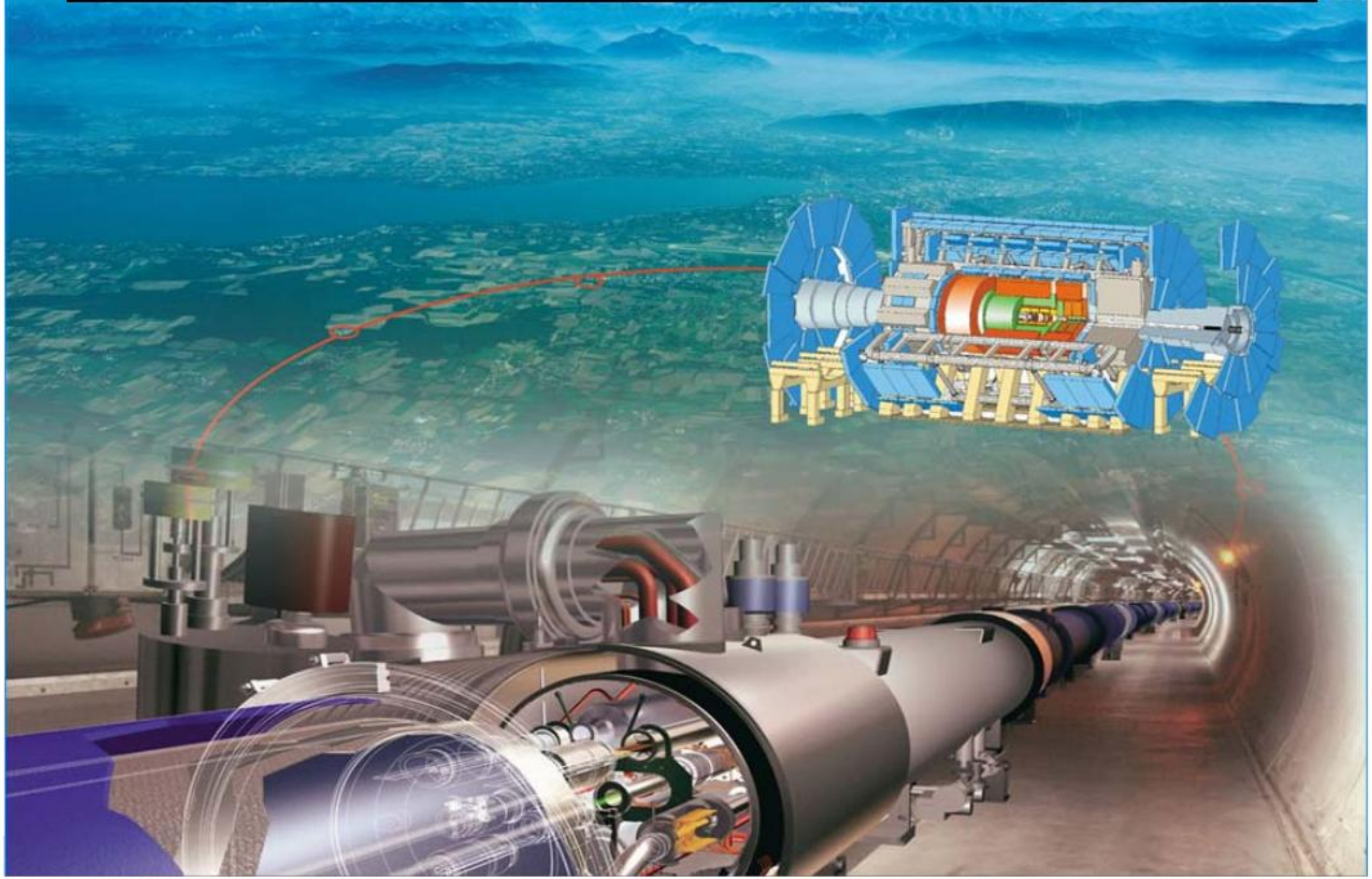
  - The delta function in the integral sends u to m_1 and hence r to:

$$r = \frac{1}{2m_1}\sqrt{m_1^4 + m_2^4 + m_3^4 - 2m_1^2 m_2^2 - 2m_1^2 m_3^2 - 2m_2^2 m_3^2} \equiv |\vec{p}_f|$$

  - The final expression of the decay rate is then given by:

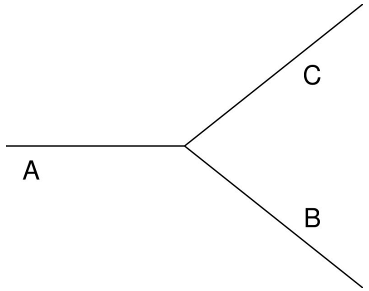
$$\Gamma = \frac{S|\vec{p}_f|}{8\pi m_1^2}|\mathcal{M}|^2$$

1.3.4 Amplitudes



Amplitudes

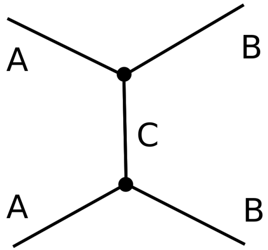
- **Calculation of amplitude $\mathcal{M}$ using a fixed set of rules ("Feynman rules")**
- **Start with introducing the methodology by studying the Feynman rules for a "toy theory":**
  - Imagine there are only three kind of particles (A, B and C) with masses m_A, m_B and m_C a spin of 0 and each is its own antiparticle.
    - There is only one vertex by which they interact:



We will assume that A is the heaviest of all three particles (it weighs more than the other two combined).

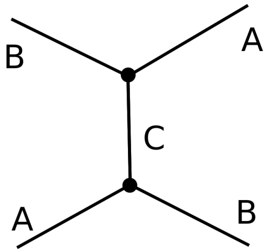
Amplitudes

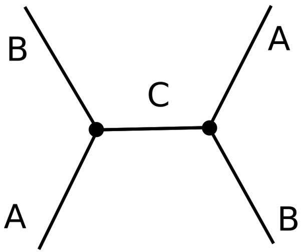
- **Feynman rules for a "toy theory":**
  - Scattering processes:
    - A + A → B + B



Lowest order diagrams

    - A + B → A + B





Amplitudes

- **To calculate $i\mathcal{M}$ use the following recipe:**
  1. **Notation:** Label the incoming and outgoing four-momenta p_1, p_2 ,..., p_n. Label the internal momenta q_1, q_2 ,... . Put an arrow beside each line, to keep track of the "positive" direction (forward in time for external lines and arbitrary for internal lines).
  2. **Vertex factor:** For each vertex write down a factor $-ig$
  3. **Propagators:** For each internal line write a factor:

     $$\frac{i}{q_j^2 - m_j^2}$$

     where q_j is the four-momentum of the line and m_j is the mass of the particle described by the internal line. (Note that virtual particles do not lie on their mass shell)
  4. **Conservation of energy and momenta:** For each vertex, write a delta function of the form:

     $$(2\pi)^4 \delta^4 (k_1 + k_2 + k_3)$$

     where the K's are the four-momenta of the three particles coming into (or out of) the vertex.
  5. **Integration over internal momenta:** For each internal line write down a factor:

     $$\frac{1}{(2\pi)^4} d^4 q_j$$

  6. **Cancel all delta functions:** $(2\pi)^4 \delta^4 (p_1 + p_2 + ... - p_n)$

Amplitudes

- **Feynman rules for a "toy theory"**
  - **Example** (Lifetime of particle A):
    - Lowest order diagram has no internal line
    - There is one vertex
      - Obtain:

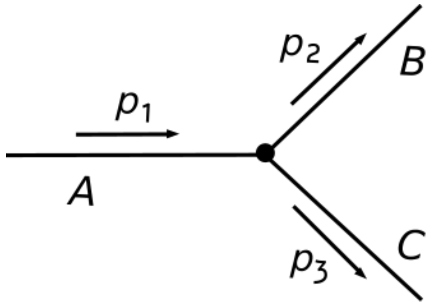
        $-ig$ (Rule 2) $(2\pi)^4\delta^4(p_1 - p_2 - p_3)$ (Rule 4)

      - Cancel the delta function (Rule 6)
    - Thus the amplitude at the lowest order is:

$$i\mathcal{M} = -ig \rightarrow \mathcal{M} = g$$

    - The decay rate and lifetime are therefore:

$$\Gamma = \frac{g^2|\vec{p}_f|}{8\pi m_A^2}$$ **and** $$\tau = \frac{1}{\Gamma} = \frac{8\pi m_A^2}{g^2|\vec{p}_f|}$$



Amplitudes

- **Feynman rules for a "toy theory"**
  - **Example** (A + A → B + B scattering):
    - Rule 1 - 5 yield:

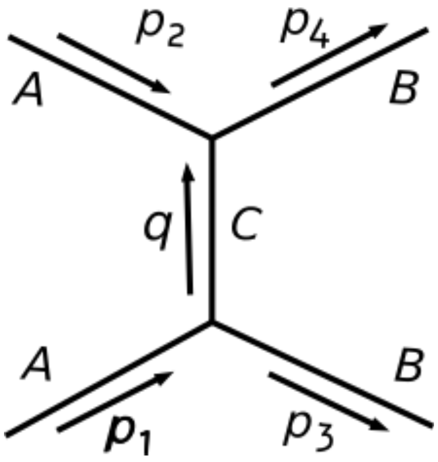
$$-i\,(2\pi)^4\,g^2\int\frac{1}{q^2-m_C^2}\delta^4\left(p_1-p3-q\right)\delta^4\left(p_2+q-p_4\right)d^4q$$

    - Doing the integral, the second delta function sends $q \to p_4 - p_2$, and we obtain

$$-ig^2\frac{1}{\left(p_4-p_2\right)^2-m_C^2}\left(2\pi\right)^4\delta^4\left(p_1+p_2-p_3-p_4\right)$$

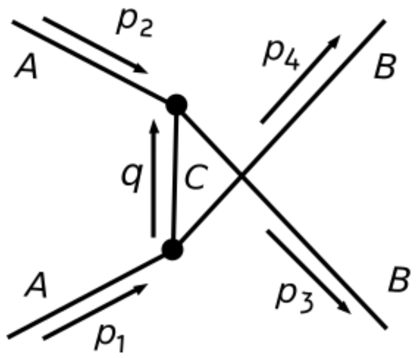
    - After applying Rule 6 we obtain:

$$\mathcal{M}=\frac{g^2}{\left(p_4-p_2\right)^2-m_C^2}$$



Amplitudes

- **Feynman rules for a "toy theory"**
  - **Example** (A + A → B + B scattering):
    - A second Feynman diagram contributes to process
      - Since the diagrams differ only by the interchange of p_3 with p_4, there is no need to compute the amplitude from scratch.
    - The total amplitude is:

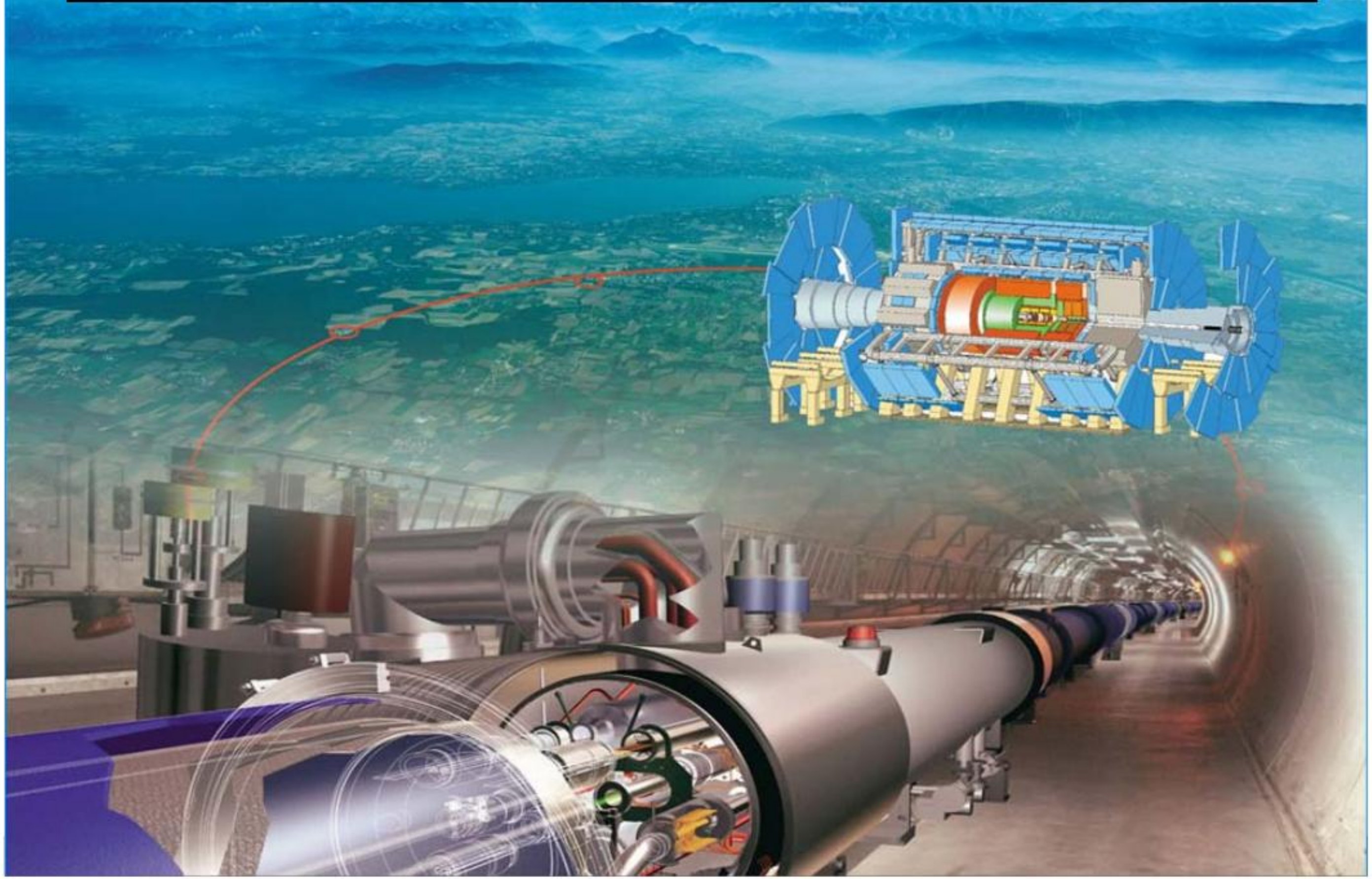


$$\mathcal{M} = \frac{g^2}{(p_4 - p_2)^2 - m_C^2} + \frac{g^2}{(p_3 - p_2)^2 - m_C^2} \Rightarrow \mathcal{M} = -\frac{g^2}{\vec{p}_f^{\,2} \sin^2\theta}$$

    - Finally, the differential cross section is:

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\left(\frac{g^2}{16\pi E \vec{p}^{\,2} \sin^2\theta}\right)^2$$

1.3.5 Feynman Rules



Feynman Rules (Tree Level)

- **External lines:**
  - **Spin 0:** (nothing)
  - **Spin ½:**
    - Incoming particle: u
    - Incoming antiparticle: $\bar{v}$
    - Outgoing particle: $\bar{u}$
    - Outgoing antiparticle: v
  - **Spin 1:**
    - Incoming: ε_μ
    - Outgoing: ε^*_μ

- **Propagators:**
  - **Spin 0:**

$$\frac{i}{q^2 - m^2}$$

  - **Spin ½:**

$$\frac{i\left(\not{q} + m\right)}{q^2 - m^2}$$

with: $\not{q} = \gamma^\mu q_\mu$

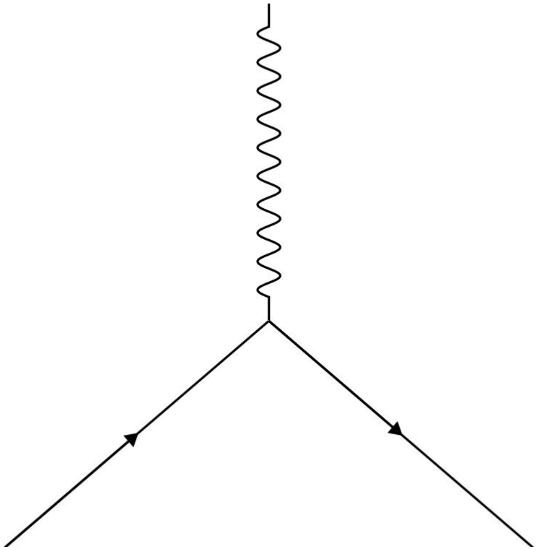
  - **Spin 1:**

$$\frac{-ig_{\mu\nu}}{q^2} \qquad \frac{-i\left[g_{\mu\nu} - q_\mu q_\nu / m^2\right]}{q^2 - m^2}$$

massless **massive**

Feynman Rules (Tree Level)

- **Vertex factors:**
  - **QED:**

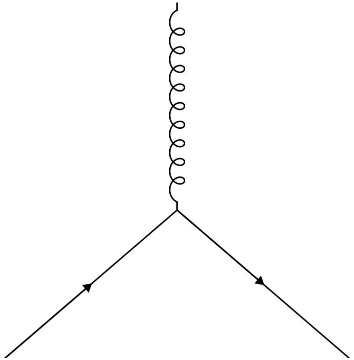


Coupling between photon and charged fermions:

$$ig_e\gamma^\mu \quad \left(g_e = \sqrt{4\pi\alpha}\right)$$

Feynman Rules (Tree Level)

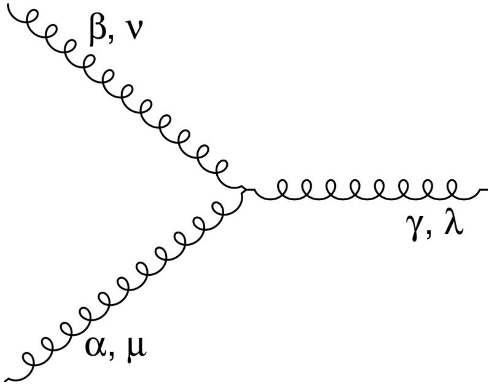
- **Vertex factors:**
  - **QCD:**

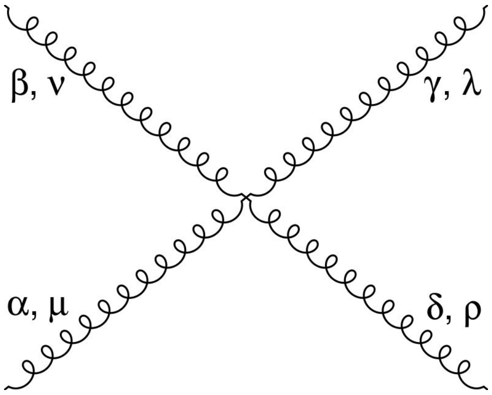


$$\frac{-ig_s}{2}\lambda^\alpha\gamma^\mu$$

with: $g_S = \sqrt{4\pi\alpha_S}$

$$-g_s f^{\alpha\beta\gamma}[g_{\mu\nu}(q_1 - q_2)_\lambda + g_{\nu\lambda}(q_2 - q_3)_\mu + g_{\lambda\mu}(q_3 - q_1)_\nu]$$





$$-ig_s^2[f^{\alpha\beta\eta}f^{\gamma\delta\eta}(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}) + f^{\alpha\delta\eta}f^{\beta\gamma\eta}(g_{\mu\nu}g_{\lambda\rho} - g_{\mu\lambda}g_{\nu\rho}) + f^{\alpha\gamma\eta}f^{\delta\beta\eta}(g_{\mu\rho}g_{\nu\lambda} - g_{\mu\nu}g_{\lambda\rho})]$$

Coupling between quarks and gluons and gluon self coupling

Feynman Rules (QCD)

- **Structure constants** **$f^{\alpha\beta\gamma}$ are defined via the commutators of the Gell-Mann matrices:**

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The first three matrices resemble the Pauli-matrices, while the next four are obtained from swapping rows and columns from λ^1, λ^2 and λ^3

Feynman Rules (QCD)

- **The commutators of the Gell-Mann matrices follow:**

$$[\lambda^\alpha, \lambda^\beta] = 2if^{\alpha\beta\gamma}\lambda^\gamma$$

- **The structure constants are completely asymmetric:**

$$f^{\beta\alpha\gamma} = f^{\alpha\gamma\beta} = -f^{\alpha\beta\gamma}$$

- **There are 8 × 8 × 8 = 512 structure constants, but most of them are zero and the rest can be worked out via the antisymmetry relation from the following set:**

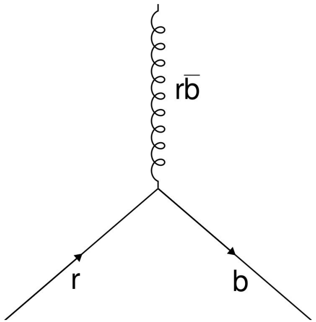
$$f^{123} = 1, \quad f^{147} = f^{246} = f^{257} = f^{345} = f^{516} = f^{637} = \frac{1}{2}$$

$$f^{458} = f^{678} = \frac{\sqrt{3}}{2}$$

Feynman Rules (QCD)

- **Colour charge:**
  - Quarks come in three colours, “red” (r), “blue” (b), and “green” (g).
  - A quark state is described by a spinor $u^{(s)}(p)$, giving its momentum and spin, and a three-element vector c giving its colour:

$$c = \begin{pmatrix}1\\0\\0\end{pmatrix} \text{ for red, } \begin{pmatrix}0\\1\\0\end{pmatrix} \text{ for blue, } \begin{pmatrix}0\\0\\1\end{pmatrix} \text{ for green}$$



  - Gluons are responsible for the transfer of colour:
    - Example (red quark turns into a blue quark (after emitting a quark)
  - Gluons carry a unit of colour and a unit of anticolour
    - In terms of colour SU(3) symmetry (on which the QCD is based) these states exist within a colour octet:

$$|1\rangle = (r\bar{b} + b\bar{r})/\sqrt{2} \qquad |5\rangle = -i(r\bar{g} - g\bar{r})/\sqrt{2}$$
$$|2\rangle = -i(r\bar{b} - b\bar{r})/\sqrt{2} \qquad |6\rangle = (b\bar{g} + g\bar{b})/\sqrt{2}$$
$$|3\rangle = (r\bar{r} - b\bar{b})/\sqrt{2} \qquad |7\rangle = -i(b\bar{g} - g\bar{b})/\sqrt{2}$$
$$|4\rangle = (r\bar{g} + g\bar{r})/\sqrt{2} \qquad |8\rangle = (r\bar{r} + b\bar{b} - 2g\bar{g})/\sqrt{6}$$

Feynman Rules (QCD)

- **Colour charge:**
  - Colour singlets such as

$$|9\rangle = \left(r\bar{r} + b\bar{b} + g\bar{g}\right)/\sqrt{3}$$

  do not exist

    - Singlets would appear as free particles in nature

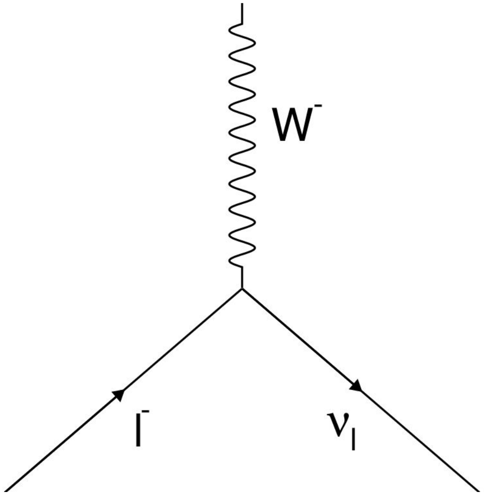
- **For external quark lines:**
  - Incoming particle: $u^{(S)}(p)c$
  - Incoming antiparticle: $\bar{v}^{(S)}(p)c^\dagger$
  - Outgoing particle: $\bar{u}^{(S)}(p)c^\dagger$
  - Outgoing antiparticle: $v^{(S)}(p)c$

with: $c^\dagger = \bar{c}^*$

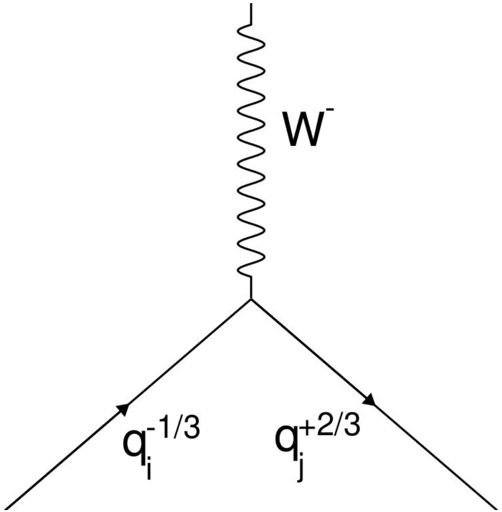
Feynman Rules (Tree Level)

- **Vertex factors:**
  - **GSW:**

Coupling between a W boson and leptons (quarks)



$$\frac{-ig_W}{2\sqrt{2}}\gamma^\mu\left(\mathbb{1}-\gamma^5\right)$$



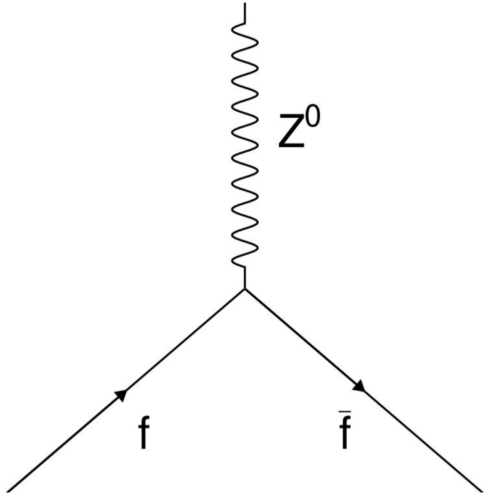
$$\frac{-ig_W}{2\sqrt{2}}\gamma^\mu\left(\mathbb{1}-\gamma^5\right)V_{ij}$$

CKM matrix element

with i = u, c or t and j = d, s or b

Feynman Rules (Tree Level)

- **Vertex factors:**
  - **GSW:**



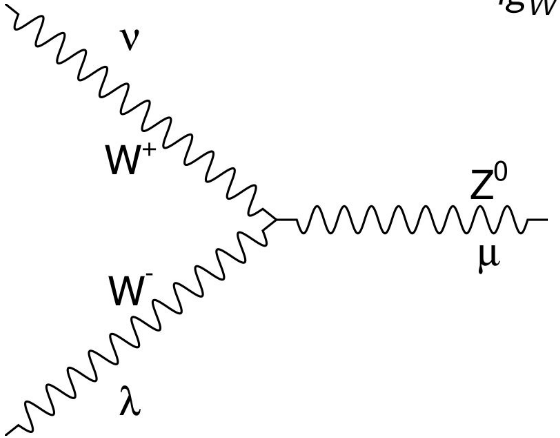
$$\frac{-ig_Z}{2}\gamma^\mu\left(c_V^f - c_A^f\gamma^5\right)$$

Coupling between a Z boson and fermions

f	c_V	c_A
ν_e, ν_μ, ν_τ	$\frac{1}{2}$	$\frac{1}{2}$
e^-, μ^-, τ^-	$-\frac{1}{2} + 2\sin^2\theta_W$	$-\frac{1}{2}$
u, c, t	$\frac{1}{2} - \frac{4}{3}\sin^2\theta_W$	$\frac{1}{2}$
d, s, b	$-\frac{1}{2} + \frac{2}{3}\sin^2\theta_W$	$-\frac{1}{2}$

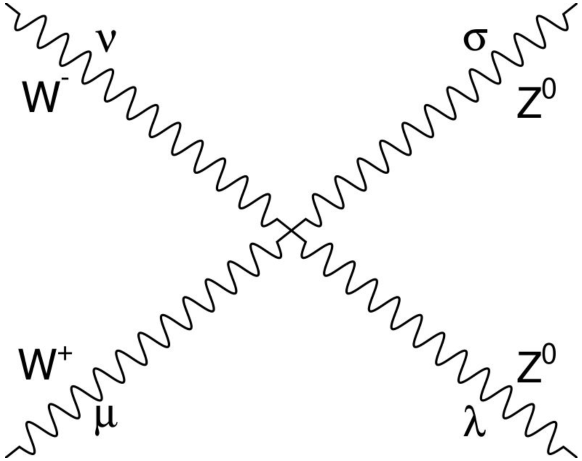
Feynman Rules (Tree Level)

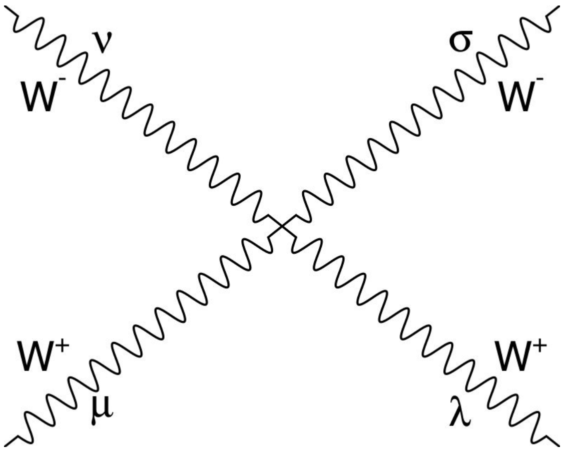
- **Vertex factors:**
  - **GSW:**



$$ig_W \cos\theta_W [g_{\nu\lambda}(q_1 - q_2)_\mu + g_{\lambda\mu}(q_2 - q_3)_\nu + g_{\mu\nu}(q_3 - q_1)_\lambda]$$

$$-ig_W^2 \cos^2\theta_W (2g_{\mu\nu}g_{\lambda\sigma} - g_{\mu\lambda}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\lambda})$$





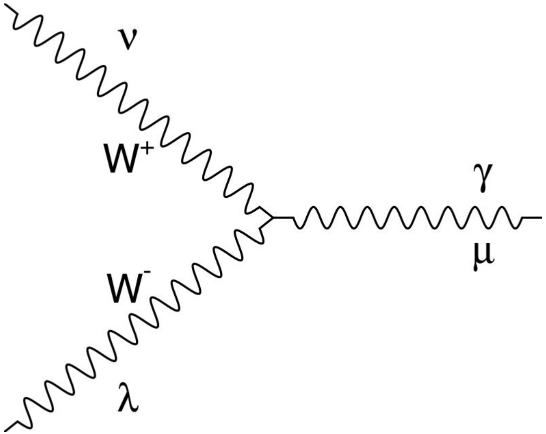
$$ig_W^2 (2g_{\mu\lambda}g_{\nu\sigma} - g_{\mu\nu}g_{\lambda\sigma} - g_{\mu\sigma}g_{\nu\lambda})$$

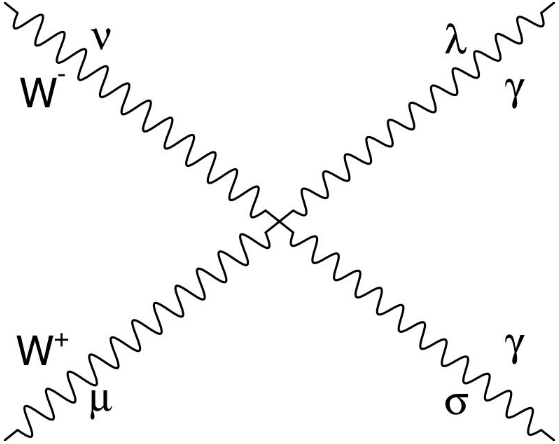
With: $g_W = \frac{g_e}{\sin\theta_W}$ **and** $g_Z = \frac{g_e}{\sin\theta_W \cos\theta_W}$

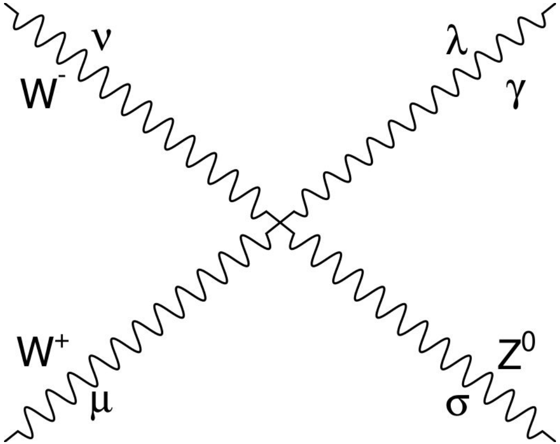
Feynman Rules (Tree Level)

- **Vertex factors:**
  - **GSW:**

$$-ig_e^2(2g_{\mu\nu}g_{\lambda\sigma} - g_{\mu\lambda}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\lambda})$$







$$ig_e[g_{\nu\lambda}(q_1 - q_2)_\mu + g_{\lambda\mu}(q_2 - q_3)_\nu + g_{\mu\nu}(q_3 - q_1)_\lambda]$$

$$-ig_e g_W \cos\theta_W(2g_{\mu\nu}g_{\lambda\sigma} - g_{\mu\lambda}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\lambda})$$

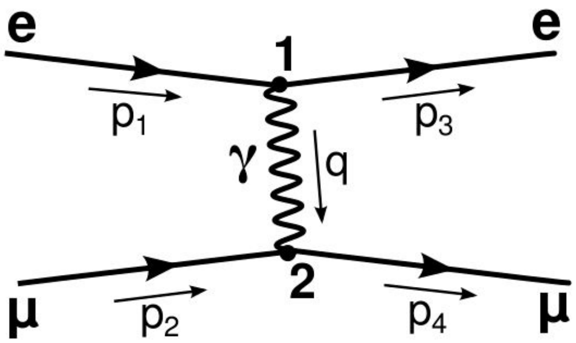
Feynman Rules (Application)

- **Example:** Calculate cross section for the process:

$$e^-(p_1, s_1) + \mu^-(p_2, s_2) \rightarrow e^-(p_3, s_3) + \mu^-(p_4, s_4)$$

where p_i and s_i denote the four momenta and spin configurations of the particles.

- Following the Feynman rules of the QED we obtain:



The arrows next to the fermion and photon lines denote the directions of the particle momenta.

$$-i\mathcal{M} = \int \bar{u}^{s_3}(p_3)(-ig_e\gamma^\mu)u^{s_1}(p_1)\frac{-ig_{\mu\nu}}{q^2}\bar{u}^{s_4}(p_4)(-ig_e\gamma^\nu)u^{s_2}(p_2)$$

$$\times (2\pi)^4 \underbrace{\delta^4(p_1 - p_3 - q)}_{*} \underbrace{(2\pi)^4\delta^4(p_2 + q - p_4)}_{**} \frac{d^4q}{(2\pi)^4}$$

- The delta function * sends q to p_1 - p_3 and ** becomes:

$$(2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4)$$

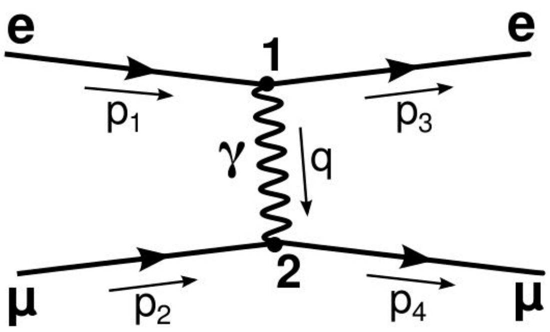
which we have to drop according to the last Feynman rule

Feynman Rules (Application)

- We therefore obtain:

$$\mathcal{M} = \frac{-g_e^2}{(p_1 - p_3)^2} \left[\bar{u}^{s_3}(p_3)\gamma^\mu u^{s_1}(p_1)\right] \left[\bar{u}^{s_4}(p_4)\gamma_\mu u^{s_2}(p_2)\right]$$

which is a number and can be calculated once the momenta p_i and spin s_i configurations are specified



- **Spin averaging:**
  - **In HEP experiments, particle beams are usually unpolarised** and detectors do not distinguish between spin states.
    - Thus measured cross sections correspond to a combination of different spin configurations.
  - **An unpolarised beam means that the probability of having the incoming electron (muon) spin in the up/down state is 50%.**
  - To obtain the unpolarised cross section one therefore has to **average over the four initial state spin configurations.**

Feynman Rules (Application)

- **Spin averaging:**
  - The fact that the detector does not distinguish between the different spin states (up, down) of the outgoing particles, means that the combinations of all possible spin final states is the final measurement
    - I.e., the sum of the processes that lead to (up, up), (down, up), (up, down), and (down, down).
  - **The matrix element/amplitude is the only part of the cross section that depends on the particle spin.**
    - **Average over the initial spin configurations and sum over the final state spin configurations:**

$$\overline{|\mathcal{M}|^2} = \underbrace{\frac{1}{4}\sum_{s_1}\sum_{s_2}}_{\text{averaging}}\underbrace{\sum_{s_3}\sum_{s_4}}_{\text{summing}}|\mathcal{M}(s_1, s_2, s_3, s_3)|^2 \quad \textbf{(13)}$$

Feynman Rules (Application)

- **Spin averaging:**
  - We now calculate $|\mathcal{M}|^2$ and use a simplified notation with $u(i) \equiv u^{s_i}(p_i)$:

$$|\mathcal{M}|^2 = \mathcal{M}\mathcal{M}^* = \frac{g_e^4}{(p_1 - p_3)^4} [\bar{u}(3)\gamma^\mu u(1)] [\bar{u}(4)\gamma_\mu u(2)] [\bar{u}(3)\gamma^\nu u(1)]^* [\bar{u}(4)\gamma_\nu u(2)]^*$$
$$= \frac{g_e^4}{(p_1 - p_3)^4} [\bar{u}(3)\gamma^\mu u(1)] [\bar{u}(3)\gamma^\nu u(1)]^* [\bar{u}(4)\gamma_\mu u(2)] [\bar{u}(4)\gamma_\nu u(2)]^* \quad \textbf{(14)}$$

  - **Casimir's trick:**
    - We encounter here twice the generic form:

$$G = [\bar{u}(a)\Gamma_1 u(b)] [\bar{u}(a)\Gamma_2 u(b)]^*$$
$$= [\bar{u}(a)\Gamma_1 u(b)] [\bar{u}(a)\Gamma_2 u(b)]^\dagger$$

Complex conjugate is the same as hermitian conjugate as the quantity in the square brackets is a 1 × 1 matrix

where Γ stands for a γ matrix.

Feynman Rules (Application)

- **Spin averaging:**
  - **Casimir's trick:**
    - We examine the second bracket:

$$
\begin{aligned}
[\bar{u}(a)\Gamma_2 u(b)]^\dagger &= \left[u^\dagger(a)\gamma^0\Gamma_2 u(b)\right]^\dagger \\
&= \left[\gamma^0\Gamma_2 u(b)\right]^\dagger u(a) \\
&= u^\dagger(b)\Gamma_2^\dagger \underbrace{\gamma^{0\dagger}}_{\gamma^0} u(a) \\
&= u^\dagger(b) \underbrace{\gamma^0\gamma^0}_{1} \Gamma_2^\dagger\gamma^0 u(a) \\
&= \bar{u}(b) \underbrace{\gamma^0\Gamma_2^\dagger\gamma^0}_{\equiv\overline{\Gamma_2}} u(a)
\end{aligned}
$$

Feynman Rules (Application)

- **Spin averaging:**
  - **Casimir's trick:**
    - With this definition, the generic form reads:

$$G = [\bar{u}(a)\Gamma_1 u(b)]\left[\bar{u}(b)\overline{\Gamma_2}u(a)\right]$$

    - Summing over the spin orientations of particle b:

$$\sum_{s_b} G = \bar{u}(a)\Gamma_1 \underbrace{\left(\sum_{s_b} u(b)\bar{u}(b)\right)}_{*} \overline{\Gamma_2}u(a)$$

where * follows the so called **completeness relation**:

$$\sum_{s_b=1}^{2} u^{s_b}(p_b)\bar{u}^{s_b}(p_b) = \gamma^{\mu}p_{b,\mu} + m_b$$

Feynman Rules (Application)

- **Spin averaging:**
  - **Casimir's trick:**
    - Thus we obtain:

$$\sum_{s_b} G = \bar{u}(a) \underbrace{\Gamma_1 \left(\not{p}_b + m_b \right) \overline{\Gamma_2}}_{\equiv Q} u(a)$$

with: $\not{p} = \gamma^\mu p_\mu$

where Q is a 4 × 4 matrix

    - We now sum over the spin configurations of particle a:

$$\sum_{s_a, s_b} G = \sum_{s_a} \bar{u}(a)\ Q\ u(a)$$

    - We write it in components so we can reorder the terms:

$$\sum_{s_a, s_b} G = \sum_{s_a=1}^{2} \sum_{\mu,\nu=1}^{4} \bar{u}_\mu^{s_a}(p_a) Q_{\mu\nu} u_\nu^{s_a}(p_a)$$

$$= \sum_{\mu,\nu} Q_{\mu\nu} \sum_{s_a} \underbrace{u_\nu^{s_a}(p_a)}_{4\times 1} \underbrace{\bar{u}_\mu^{s_a}(p_a)}_{1\times 4} = \sum_{\mu,\nu} Q_{\mu\nu} \left[\sum_{s_a} u^{s_a}(p_a) \bar{u}^{s_a}(p_a) \right]_{\nu\mu}$$

Feynman Rules (Application)

- **Spin averaging:**
  - **Casimir's trick:**
    - Exploiting again the **completeness relation**, leads to:

$$\sum_{s_a,s_b} G = \sum_{\mu,\nu} Q_{\mu\nu}\left[\not{p}_a + m_a\right]_{\nu\mu}$$
$$= \sum_{\mu}\left[Q(\not{p}_a + m_a)\right]_{\mu\mu} = \mathrm{Tr}\left(Q(\not{p}_a + m_a)\right)$$

4 × 4 matrix

    - Inserting the definitions of G and Q, we have just proven the following relation (which is referred to as **Casimir's trick**):

$$\sum_{s_a,s_b} [\bar{u}(a)\Gamma_1 u(b)]\,[\bar{u}(a)\Gamma_2 u(b)]^* = \mathrm{Tr}\left(\Gamma_1(\not{p}_b + m_b)\overline{\Gamma_2}(\not{p}_a + m_a)\right)$$

    - Once the summation over all spins is done, all that remains is to multiply matrices and calculate the trace.

Feynman Rules (Application)

- **Calculation of the spin-averaged cross section:**
  - To apply Casimir's trick to our case, we use:

$$\Gamma_1 = \gamma^\mu \qquad \Gamma_2 = \gamma^\nu \qquad \text{and} \qquad \overline{\Gamma_2} = \gamma^0 \gamma^{\nu\dagger} \gamma^0 = \gamma^\nu$$

  - Including equation (14) into equation (13) gives:

$$\begin{aligned}\overline{|\mathcal{M}|^2} &= \frac{1}{4} \sum_{s_1, s_2, s_3, s_4} \frac{g_e^4}{(p_1 - p_3)^4} \left[\bar{u}(3)\gamma^\mu u(1)\right]\left[\bar{u}(3)\gamma^\nu u(1)\right]^* \left[\bar{u}(4)\gamma_\mu u(2)\right]\left[\bar{u}(4)\gamma_\nu u(2)\right]^* \\ &= \frac{g_e^4}{4(p_1 - p_3)^4} \mathrm{Tr}\left(\gamma^\mu(\not{p}_1 + m_1)\gamma^\nu(\not{p}_3 + m_3)\right) \mathrm{Tr}\left(\gamma_\mu(\not{p}_2 + m_2)\gamma_\nu(\not{p}_4 + m_4)\right)\end{aligned} \qquad \textbf{(15)}$$

  - Denote the electron mass with m and the muon mass with M:

$$m_1 = m_3 = m_e \equiv m$$
$$m_2 = m_4 = m_\mu \equiv M$$

Feynman Rules (Application)

- **Calculation of the spin-averaged cross section:**
  - To calculate the traces from equation **(15)** we use the **trace theorems:**
    - Full details given e.g. in: D. Griffiths, **Introduction to Elementary Particles,** WILEY-VCH**,** 2008, 2nd edition, page 252-253.
    - For our purposes:
      - Rule 1:

$$\mathrm{Tr}(A+B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$$

      - Rule 2:

$$\mathrm{Tr}(\alpha A) = \alpha\,\mathrm{Tr}(A)$$

      - Rule 3:

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$$

      - Rule 10: The trace of the product of an odd number of gamma matrices is zero
      - Rule 12:

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$$

      - Rule 13:

$$\mathrm{Tr}(\gamma^\mu\gamma^\lambda\gamma^\nu\gamma^\sigma) = 4\left(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}\right)$$

Feynman Rules (Application)

- **Calculation of the spin-averaged cross section:**
  - First we will calculate the electron trace:

$$\begin{aligned}\mathrm{Tr}\left(\gamma^\mu(\not{p}_1+m)\gamma^\nu(\not{p}_3+m)\right) &= \mathrm{Tr}\left((\gamma^\mu\not{p}_1\gamma^\nu+m\gamma^\mu\gamma^\nu)(\not{p}_3+m)\right)\\ &= \mathrm{Tr}\left(\gamma^\mu\not{p}_1\gamma^\nu\not{p}_3+m\gamma^\mu\gamma^\nu\not{p}_3+m\gamma^\mu\not{p}_1\gamma^\nu+m^2\gamma^\mu\gamma^\nu\right)\\ &= \underbrace{\mathrm{Tr}\left(\gamma^\mu\not{p}_1\gamma^\nu\not{p}_3\right)}_{*}+m\underbrace{\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\not{p}_3\right)}_{=0}+m\underbrace{\mathrm{Tr}\left(\gamma^\mu\not{p}_1\gamma^\nu\right)}_{=0}+m^2\underbrace{\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\right)}_{=4g^{\mu\nu}}\end{aligned}$$

Due to rule 10 and $\not{p}=\gamma^\mu p_\mu$

Due to rule 12

    - For * we resolve the slash notation and apply Rule 13:

$$\begin{aligned}\mathrm{Tr}\left(\gamma^\mu\not{p}_1\gamma^\nu\not{p}_3\right) &= \mathrm{Tr}\left(\gamma^\mu\gamma^\lambda p_{1,\lambda}\gamma^\nu\gamma^\sigma p_{3,\sigma}\right)\\ &= p_{1,\lambda}p_{3,\sigma}\underbrace{\mathrm{Tr}(\gamma^\mu\gamma^\lambda\gamma^\nu\gamma^\sigma)}_{\text{Apply Rule 13}}\\ &= 4(p_1^\mu p_3^\nu - g^{\mu\nu}\underbrace{p_1^\sigma p_{3,\sigma}}_{p_1\cdot p_3}+p_3^\mu p_1^\nu)\end{aligned}$$

Feynman Rules (Application)

- **Calculation of the spin-averaged cross section:**
  - The muon trace results from the electron trace by replacing m with M, lowering the Greek indices, and replacing $p_1 \rightarrow p_2$ and $p_3 \rightarrow p_4$, so that equation (15) becomes:

$$\frac{4(p_1 - p_3)^4}{16 g_e^4}\overline{|M|^2} = \left(p_1^\mu p_3^\nu - g^{\mu\nu} p_1 p_3 + p_3^\mu p_1^\nu + m^2 g^{\mu\nu}\right)$$

$$\times \left(p_{2,\mu} p_{4,\nu} - g_{\mu\nu} p_2 \cdot p_4 + p_{4,\mu} p_{2,\nu} + M^2 g_{\mu\nu}\right)$$

$$= (p_1 \cdot p_2)(p_3 \cdot p_4) - (p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) + M^2 p_1 \cdot p_3$$

$$- (p_2 \cdot p_4)(p_1 \cdot p_3) + \underbrace{g^{\mu\nu} g_{\mu\nu}}_{=4}(p_1 \cdot p_3)(p_2 \cdot p_4) - (p_1 \cdot p_3)(p_2 \cdot p_4) - 4M^2 p_1 \cdot p_3$$

$$+ (p_3 \cdot p_2)(p_1 \cdot p_4) - (p_3 \cdot p_1)(p_2 \cdot p_4) + (p_3 \cdot p_4)(p_1 \cdot p_2) + M^2 p_1 \cdot p_3$$

$$+ m^2 p_2 \cdot p_4 - 4m^2 p_2 \cdot p_4 + m^2 p_4 \cdot p_2 + 4m^2 M^2$$

$$= 2(p_1 \cdot p_2)(p_3 \cdot p_4) + 2(p_1 \cdot p_4)(p_2 \cdot p_3) + 4m^2 M^2 - 2M^2 p_1 \cdot p_3 - 2m^2 p_2 \cdot p_4$$

Feynman Rules (Application)

- **Calculation of the spin-averaged cross section:**
  - The final form of the spin-averaged squared amplitude is:

$$\overline{|\mathcal{M}|^2} = \frac{8g_e^4}{(p_1 - p_3)^4}\left[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) + 2m^2M^2 - M^2 p_1 \cdot p_3 - m^2 p_2 \cdot p_4\right]$$

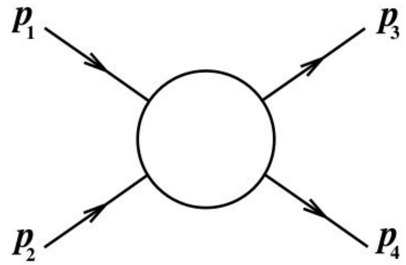
  - The cross section for unpolarised $e^- \mu^- \to e^- \mu^-$ scattering is therefore given by:

$$\frac{d\sigma}{d\Omega^*} = \frac{1}{(8\pi)^2 (p_1 + p_2)^2}\overline{|\mathcal{M}|^2}$$

1.3.6 Mandelstam variables

Mandelstam variables

- **For 2 → 2 scattering the following Mandelstam variables are defined:**



$$s \equiv (p_1 + p_2)^2 \qquad t \equiv (p_1 - p_3)^2 \qquad u \equiv (p_1 - p_4)^2$$

which are related by:

$$s + t + u = \sum_{i}^{N=4} m_i^2$$

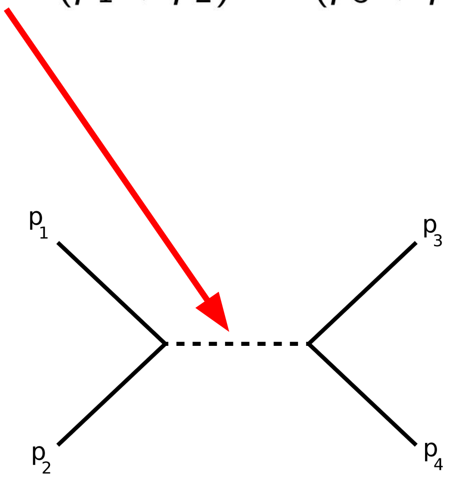
- **When neglecting the electron and muon masses, the amplitude for e-μ scattering can be written using the Mandelstam variables as:**

$$\overline{|\mathcal{M}|^2} = 2g_e^4 \frac{s^2 + u^2}{t^2}$$

Mandelstam variables

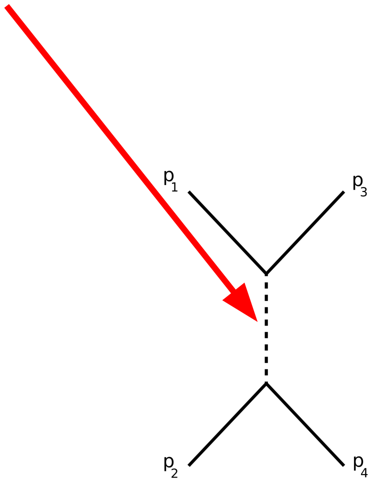
- **For 2 → 2 scattering the following Mandelstam variables are defined:**

$$s \equiv (p_1 + p_2)^2 = (p_3 + p_4)^2$$



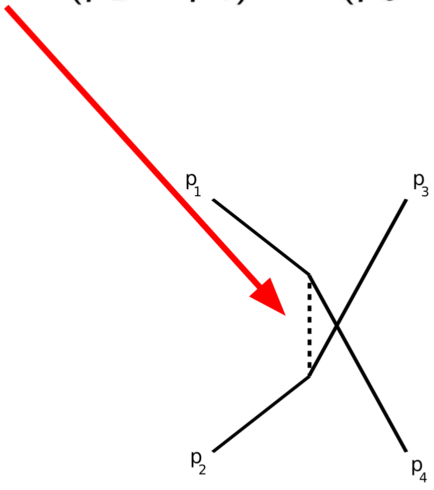
s-channel diagram

$$t \equiv (p_1 - p_3)^2 = (p_4 - p_2)^2$$



t-channel diagram

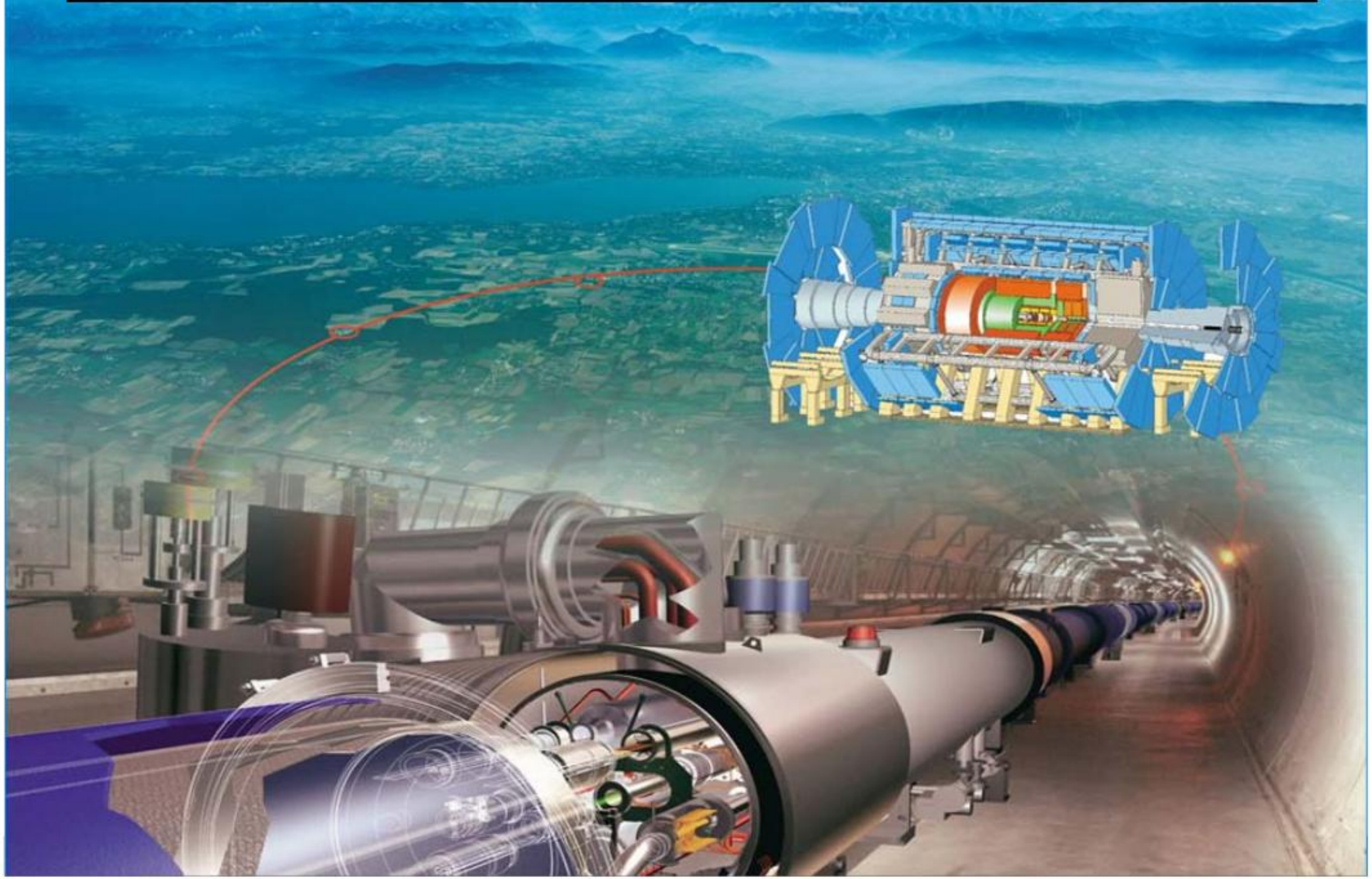
$$u \equiv (p_1 - p_4)^2 = (p_3 - p_2)^2$$



u-channel diagram

s, t, u equal the squared four-momenta of exchange particles

1.3.7 Crossing symmetry



Crossing symmetry

- **Suppose that a reaction of the form**

$$A + B \rightarrow C + D$$

is know to occur. Any of these particles can be “crossed” over to the other side of the reaction, provided it is turned into its antiparticle and the resulting interaction will also be (dynamically) allowed

  - For example:

$$A \rightarrow \bar{B} + C + D$$

$$A + \bar{C} \rightarrow \bar{B} + D$$

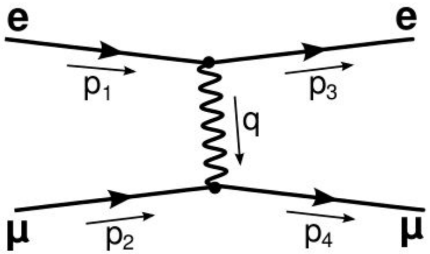
$$\bar{C} + \bar{D} \rightarrow \bar{A} + \bar{B}$$

- **This means that e.g. Compton scattering and pair annihilation are “basically” the same:** $\gamma + e^- \rightarrow \gamma e^-$ **and** $e^- + e^+ \rightarrow \gamma\gamma$
  - The amplitudes of the original diagram and the crossed diagram can easily be inferred from each other

Crossing symmetry

- **Example:**

$e^- \mu^- \rightarrow e^- \mu^-$



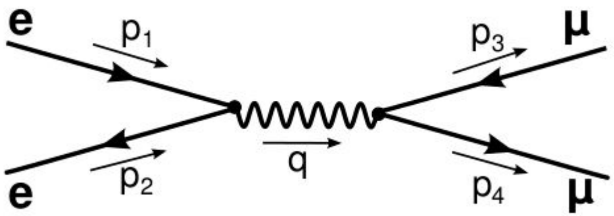
t-channel diagram

$$t = (p_1 - p_3)^2 = q^2$$

in massless limit:

$$\overline{|\mathcal{M}|^2} = 2g_e^4 \frac{s^2 + u^2}{t^2}$$

$e^+ e^- \rightarrow \mu^+ \mu^-$



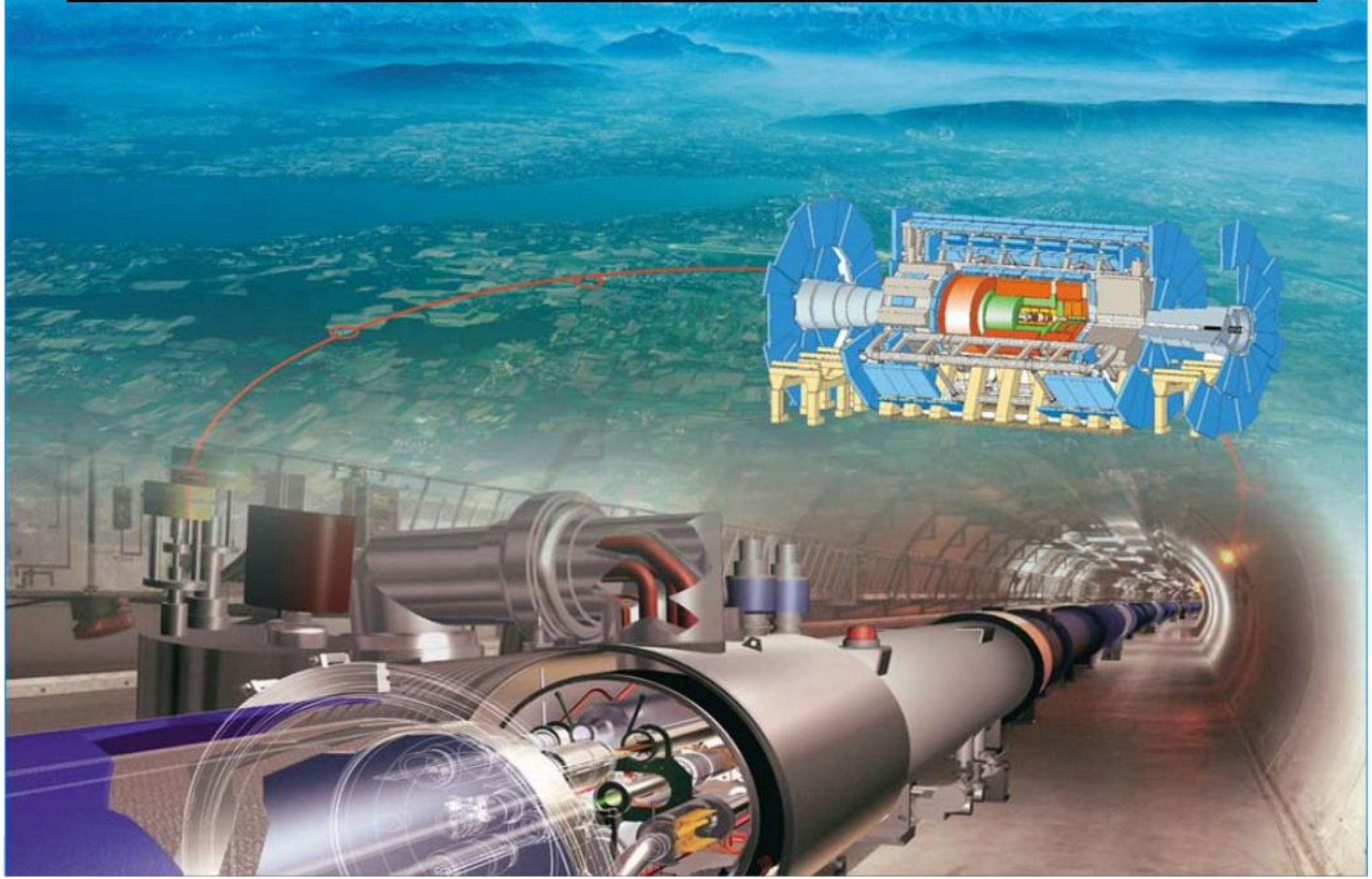
s-channel diagram

$$s = (p_1 + p_2)^2 = q^2$$

in massless limit:

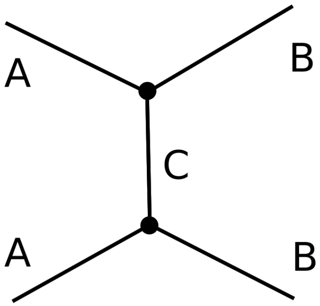
$$\overline{|\mathcal{M}|^2} = 2g_e^4 \frac{t^2 + u^2}{s^2}$$

1.3.8 Higher order corrections

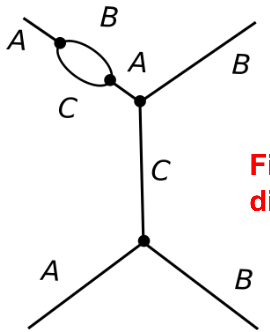


Loop diagrams

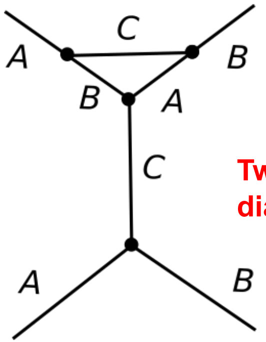
- **Back to the Feynman rules for our “toy theory”:**
  - Previously we only considered leading order diagrams to the **A + A → B + B** scattering:



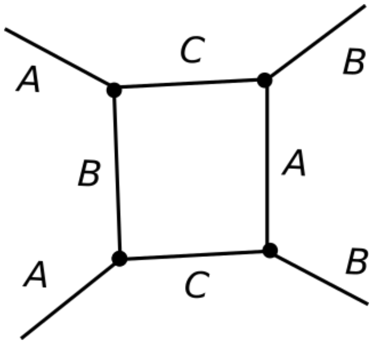
  - Several next-to-leading order (NLO) diagrams exists:



Five “self-energy” diagrams



Two “vertex correction” diagrams



One “box” diagram

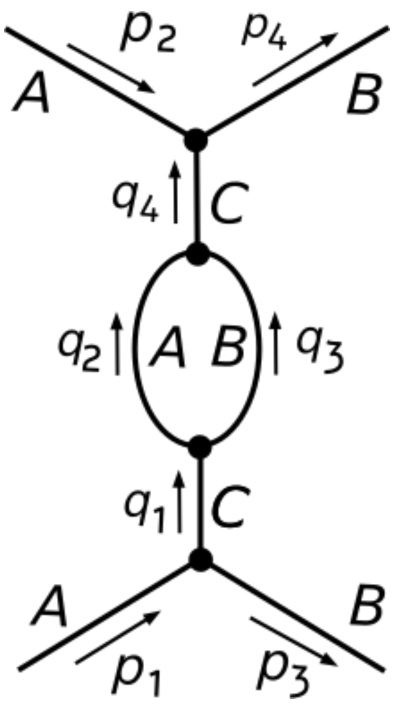
Loop diagrams

- **Back to the Feynman rules for our "toy theory":**
  - Calculate amplitude for exemplary NLO diagram:
    - Applying Feynman rules 1-5 yields:

$$g^4 \int \frac{\delta^4(p_1 - q_1 - p_3)\delta^4(q_1 - q_2 - q_3)\delta^4(q_2 + q_3 - q_4)\delta^4(q_4 + p_2 - q_4)}{(q_1^2 - m_C^2)(q_2^2 - m_A^2)(q_3^2 - m_B^2)(q_4^2 - m_C^2)} \times d^4 q_1 \, d^4 q_2 \, d^4 q_3 \, d^4 q_4$$

    - Integration over q_1, using the first delta function, replaces q_1 by $(p_1 - p_3)$, while integration over q_4, using the last delta function, replaces q_4 by $(p_4 - p_2)$:

$$\frac{g^4}{[(p_1 - p_3)^2 - m_C^2]\,[(p_4 - p_2)^2 - m_C^2]} \times \int \frac{\delta^4(p_1 - p_3 - q_2 - q_3)\delta^4(q_2 + q_3 - p_4 + p_2)}{(q_2^2 - m_A^2)(q_3^2 - m_B^2)} d^4 q_2 \, d^4 q_3$$



Loop diagrams

- **Back to the Feynman rules for our "toy theory":**
  - Calculate amplitude for exemplary NLO diagram:
    - Here, the first delta function will send $q_2 \rightarrow p_1 - p_3 - q_3$, and the second delta function becomes

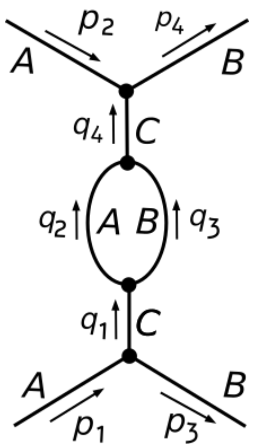
$$\delta^4(p_1 + p_2 - p_3 - p_4)$$

      which, by rule 6 has to be erased.
    - Therefore the amplitude becomes:

$$\mathcal{M} = i\left(\frac{g}{2\pi}\right)^4 \frac{1}{[(p_1 - p_3)^2 - m_C^2]^2} \int \frac{1}{[(p_1 - p_3 - q_3)^2 - m_A^2]\,(q_3^2 - m_B^2)} d^4 q_3$$

    - Trying to solve the integral (switching to spherical coordinates) will fail:

$$\int^{\infty} \frac{1}{q^4} q^3 dq = \ln q|^{\infty} = \infty$$



with:

$d^4 q = q^3 dq\, d\Omega$

Loop diagrams

- Renormalization:
  - The problem is solved by introducing a cutoff mass M:

$$\int_m^M \frac{dq}{q} = \ln \frac{M}{m}$$

  - The cutoff mass is assumed to be very large and will be taken to infinity at the end of the calculation ($M \to \infty$)
  - The introduction of the cutoff has two consequences:
    - The physical masses and couplings (i.e. what we measure) are not identical with the expressions that appear in the original Feynman rules:

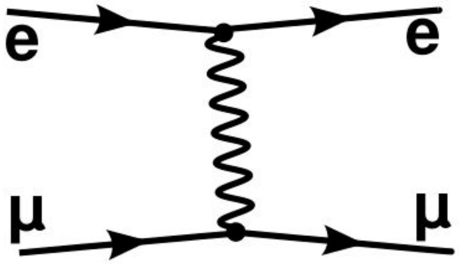
$$m_{\text{physical}} = m + \delta m$$
$$g_{\text{physical}} = g + \delta g$$

δm and δg are infinite (in the limit M → ∞) which is not catastrophic (as we will not measure them).

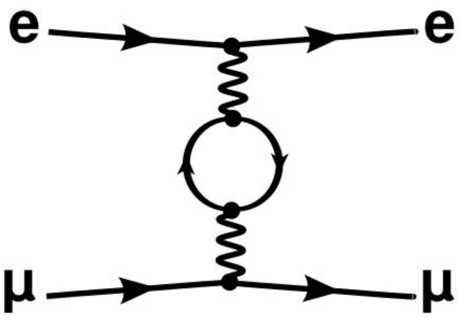
      Infinites will be taken into account as the *physical values* of m and g will be used in the Feynman rules (instead of their "bare" values)
    - The effective mass and coupling become depend on the energies of the involved particles (we speak of *"running mass"* and *"running coupling"*):
      - For more details see e.g.: D. Griffiths, **Introduction to Elementary Particles,** WILEY-VCH, 2008, 2nd edition, page 264-265

Higher order corrections to QED processes

- **Loop diagram contributions:**
  - The QED scattering matrix is expanded in terms of α/π and each Feynman diagram is a term in this expansion
    - The expansion series converges early because the coupling factor ($\alpha \approx 1/137$) is small
    - **Usually only a few diagrams are necessary to obtain a prediction with uncertainties comparable to the measurement precision of HEP**
  - Higher order Feynman diagrams have more internal lines and vertices
    - The photon can split into a fermion-antifermion pair which subsequently recombines (***vacuum polarisation***)
  - *Loop diagrams and tree level diagrams lead to the same final particle state*
    - *Thus the amplitudes have to be added and then squared to obtain the total cross section.*
  - ***The amplitude of the loop diagram is proportional to α^2***
    - *Thus the total cross section receives only a small correction from the loop diagram.*



$\sim \alpha_{\mathrm{EM}}$



$\sim \alpha_{\mathrm{EM}}^2$

Higher order corrections to QED processes

- **Renormalisation (QED):**
  - The “renormalized” coupling constant of the QED is defined via:

$$g_R \equiv g_e \sqrt{1 - \frac{g_e^2}{12\pi} \ln\left(\frac{M^2}{m^2}\right)}$$

  - The energy dependence (expressed via the correction function $f(Q^2/m^2)$), is also absorbed into the couplings constant.
    - This leads to:

$$g_{\text{eff}}(Q^2) \equiv g_R \sqrt{1 + \frac{g_R^2}{12\pi} f\left(\frac{Q^2}{m^2}\right)}$$

    where Q^2 is the photon virtuality $-q^2$ (and q is the 4-momentum of the virtual photon)
    - Which translates to:

$$\alpha(Q^2) = \alpha \sqrt{1 + \frac{\alpha}{3\pi} f\left(\frac{Q^2}{m^2}\right)} \qquad \textbf{(16)}$$

Higher order corrections to QED processes

- **Renormalisation (QED):**
  - Summing over all order (and considering all possible particles in the loop) leads to:

$$\alpha(Q^2) = \frac{\alpha(\mu^2)}{1 - \frac{\alpha(\mu^2)}{\pi} \ln \frac{Q^2}{\mu^2}}$$

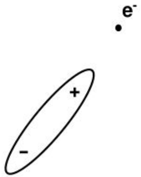
where μ is a (mass) scale.

→ **running coupling constant of the QED:** $\alpha(0) \approx \frac{1}{137}$ **and** $\alpha(Q^2 = m_Z^2) \approx \frac{1}{129}$

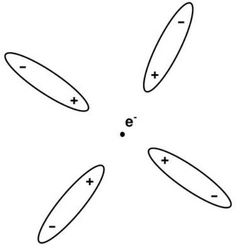
- **Measurement:**
  - The running of α can be determined for example, by measuring the cross section for e e → μ μ which is proportional to α^2. The photon virtuality Q^2 is given by s (i.e. the square of the centre-of-mass energy). By changing s through adjustments of the positron and electron beam energies, α can be determined as a function of Q^2.

Higher order corrections to QED processes

- **Physics interpretation of the Q^2 dependence of α_{QED} :**
  - Vacuum polarisation leads to a shielding of the electric charge
    - Fermion-antifermion pairs are produced in loop-diagrams by the exchanged photon.

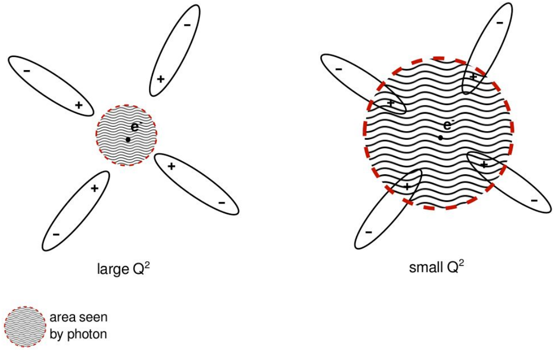


In one loop diagrams, the fermion pair forms an electric dipole



At higher orders, corresponding to several loops, several dipoles are formed:

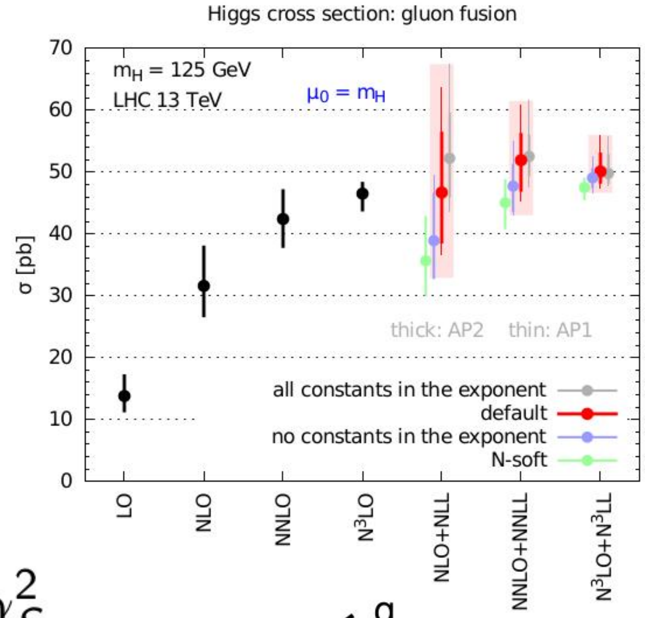
- At high Q^2 , the photon resolves the bare electron charge, while at low Q^2, the photon “sees” the charge in a larger area and part of the electron charge is shielded by the dipoles:

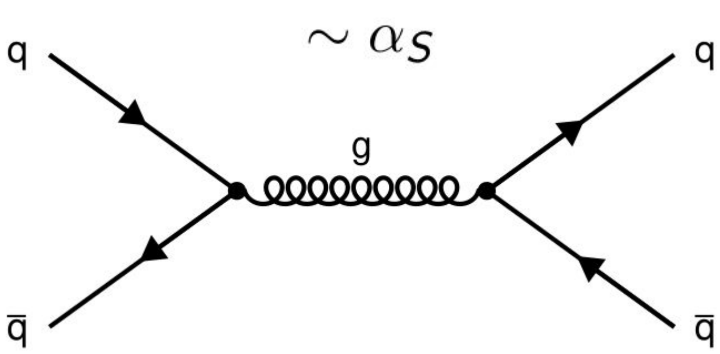


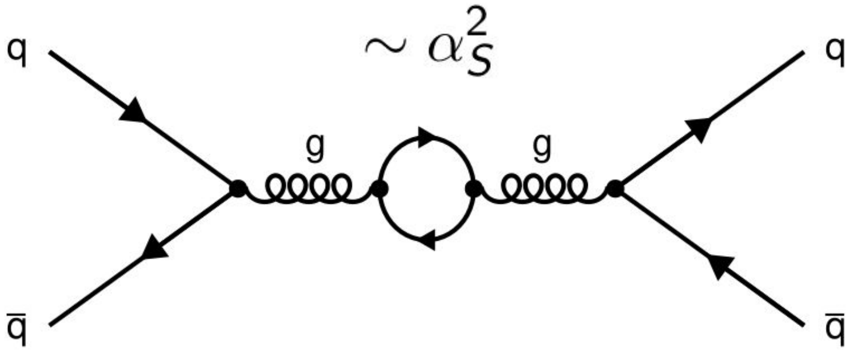
→ The electron charge seen by the photon at low Q^2 is smaller than the bare charge.

Loop diagrams

- **QCD corrections:**
  - **Significantly larger than QED corrections due to size of QCD coupling constant: $\alpha_s(m_Z) \approx 0.12$**
    - Higher order Feynman diagrams still lead to crucial contributions







Running coupling (QCD)

- **In QCD, study the scattering between quarks via exchange of gluons**
  - Analogous to photon, virtual gluon creates quark- and gluon-loops
- **Analogous to equation (13), the quark- and gluon-loop contribution lead to:**

$$\left[\alpha_S(Q^2)\right]_{q\bar{q}} = \alpha_S(\mu^2)\left(1 + N_f \frac{\alpha_S(\mu^2)}{6\pi} \ln\left(\frac{Q^2}{\mu^2}\right)\right)$$

and

$$\left[\alpha_S(Q^2)\right]_{gg} = \alpha_S(\mu^2)\left(1 - 11 \frac{\alpha_S(\mu^2)}{4\pi} \ln\left(\frac{Q^2}{\mu^2}\right)\right)$$

- **Summing both expressions (considering all orders) finally gives:**

$$\alpha_S(Q^2) = \frac{12\pi}{(33 - 2N_f)\ln\left(\frac{Q^2}{\Lambda^2}\right)}$$

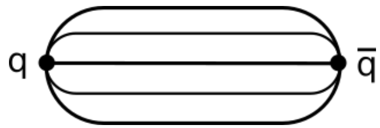
with:

$$\Lambda^2 = \mu^2 \exp\left(-\frac{12\pi}{(33 - 2N_f)\alpha_S(\mu^2)}\right)$$

which is applicable for $Q^2 \gg \Lambda^2$ (Λ is the non-perturbative scale of QCD)
  - $\Lambda \sim 10^2$ MeV (measured in e^+e^- collisions)
- **The strong coupling constant α_S decreases with increasing Q^2**

Colour Confinement

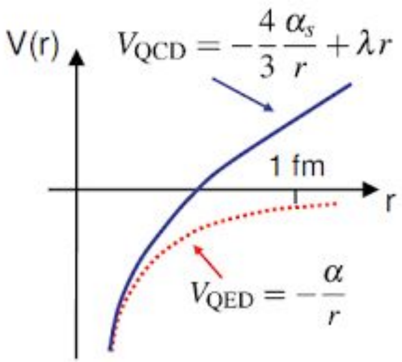
- For small distances (í.e. large energies) r << R^{Proton}
  - The potential between two quarks is ~ 1/r (analogous to the Coulomb-Potential) as gluons are massless
    - Different behaviour (wrt. QED) for large(r) distances between charges
- Potential has also term that increases linearly
  - **All field lines go from one quark to the other**
- **Potential:**



$$V(r) = -\frac{4\alpha_S}{3r} + \sigma \cdot r$$

with $\sigma \approx 0.9$ GeV/fm

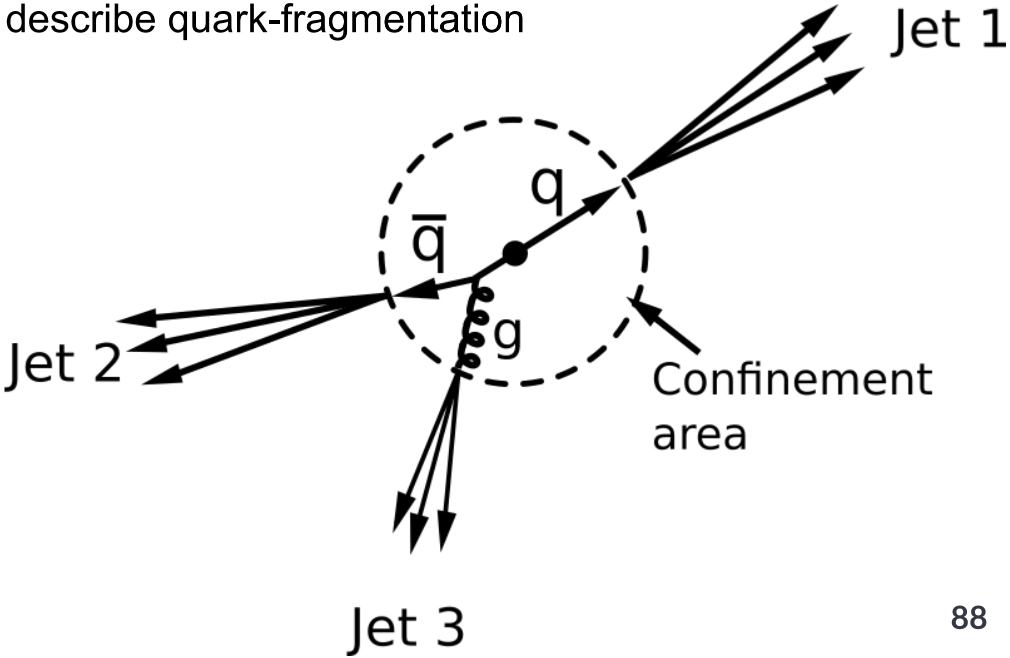
“string tension”



- **Confinement:**
  - V → ∞ for r→ ∞
- **Asymptotic freedom:**
  - V → 0 for r→ 0 (q^2→ ∞)

Colour Confinement

- **If the distance between two colour charges exceeds values above the order of 1 fm, it becomes energetically favorable for a new quark–antiquark pair to be created from the vacuum , rather than extending the tube further.**
  - Described via **Hadronization** process**:**
    - Use phenomenological models to describe quark-fragmentation
      - **Field-Feynman model**
      - **Lund-model**

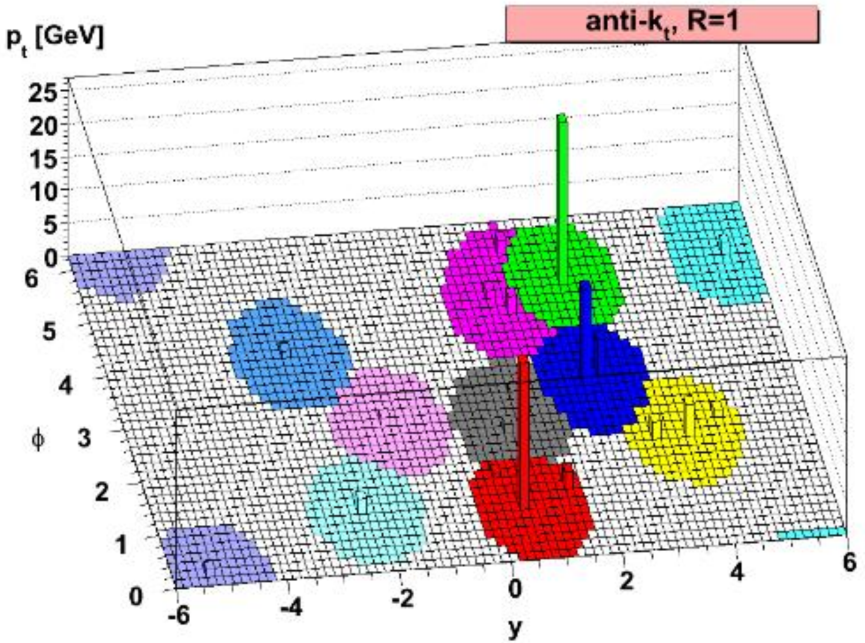


Jets

- **Jets:** Collimated bunches of stable hadrons, originating from partons (quarks and gluons) after fragmentation and hadronization
- Require collinear- and infrared-safety i.e. jets are unchanged by:
    - Collinear splitting
    - Soft emissions
- LHC experiments preferrably use so called **sequential clustering algorithms**
- Application: Calculate for all pairs of particles i an j:

$$d_{ij} = \min(k_{i,T}^{2p}, k_{j,T}^{2p}) \frac{\Delta_{ij}^2}{R^2}$$

$$d_{iB} = k_{i,T}^{2p}$$



The pair with the smallest d_{ij} is clustered if $d_{ij} < d_{iB}$, for $d_{iB} < d_{ij}$ object i is called a jet

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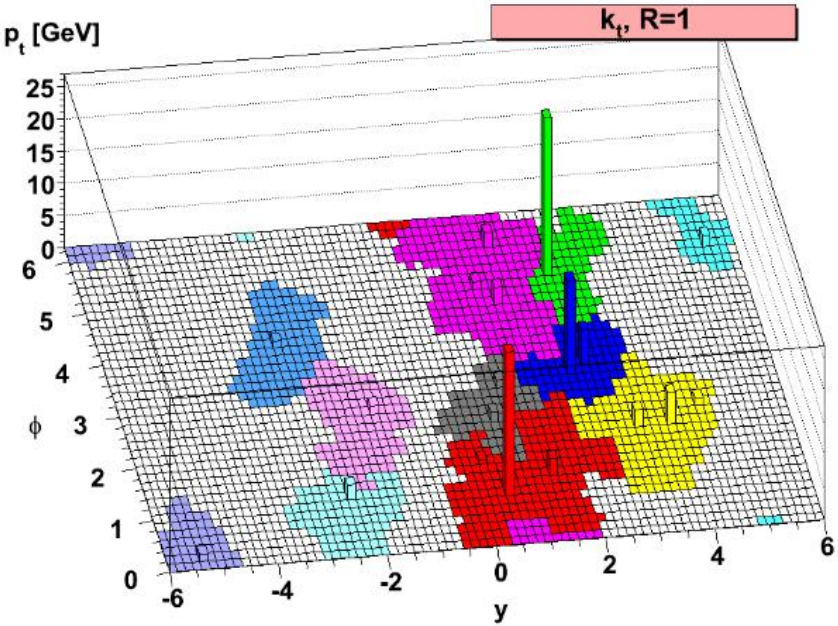

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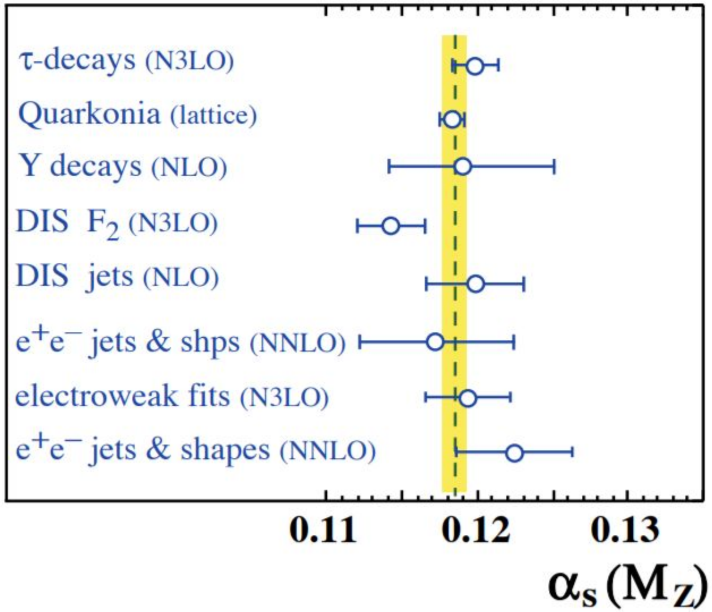
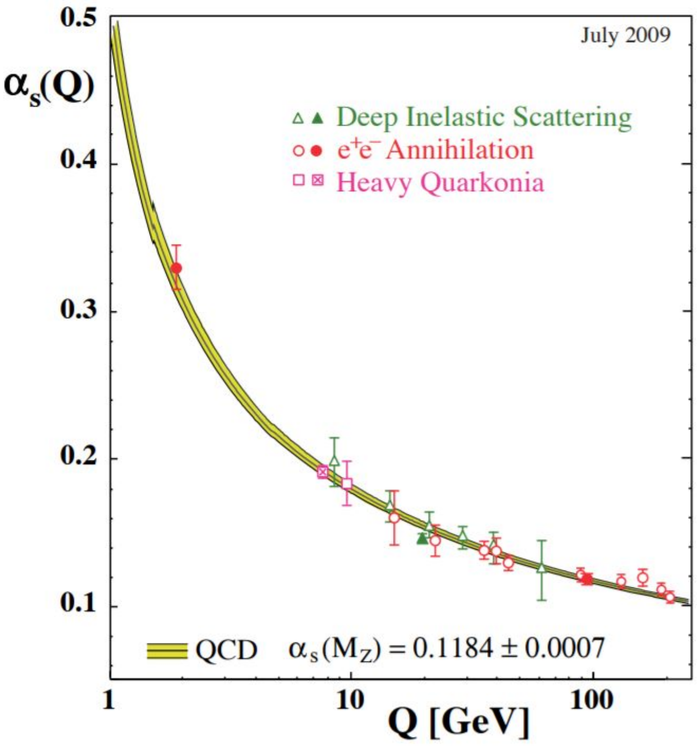
Measurements of the strong coupling constant

- Strong coupling constant can be determined via production cross section measurement of hadrons in e^+e^- collisions:

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum_j Q_j^2 \cdot \left(1 + \left(\frac{\alpha_S}{\pi}\right) + 1.4\left(\frac{\alpha_S}{\pi}\right)^2 - 12.8\left(\frac{\alpha_S}{\pi}\right)^3 + \ldots\right)$$





From: https://pdg.lbl.gov/2010/reviews/rpp2010-rev-qcd.pdf

QCD scales

- **Exemplary choices of QCD scales:**
  - **Fixed scale:**

    ~ mass of particle under study

  - **Dynamic scale** (for multi particle final states):

$$Q = \frac{1}{2}\sum_i \underbrace{\sqrt{m_i^2 + p_{\mathrm{T,i}}^2}}_{m_{\mathrm{T,i}}}$$